Confluence algebras and acyclicity of the Koszul complex

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Abstract. The N-Koszul algebras are N-homogeneous algebras satisfying a homological property. These algebras are characterised by their Koszul complex: an N-homogeneous algebra is N-Koszul if and only if its Koszul complex is acyclic. Methods based on computational approaches were used to prove N-Koszulness: an algebra admitting a side-confluent presentation is N-Koszul if and only if the extra-condition holds. However, in general, these methods do not provide an explicit contracting homotopy for the Koszul complex. In this article we present a way to construct such a contracting homotopy. The property of side-confluence enables us to define specific representations of confluence algebras. These representations provide a candidate for the contracting homotopy. When the extra-condition holds, it turns out that this candidate works. We make explicit our construction on several examples.

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1 Introduction

An overview on Koszulness and N-Koszulness

Koszul algebras. *Koszul algebras* were defined by Priddy in [Pri70] as *quadratic algebras* which satisfy a homological property. A quadratic algebra is a graded associative algebra over a field $\mathbb{K}$ which admits a *quadratic presentation* $\langle X \mid R\rangle$, that is, X is a set of generators and R is a set of quadratic relations. If $\mathbf{A}$ is a quadratic algebra, the field $\mathbb{K}$ admits a left and right $\mathbf{A}$-module structure induced by the $\mathbb{K}$-linear projection $\varepsilon : \mathbf{A} \longrightarrow \mathbb{K}$ which maps any generator of $\mathbf{A}$ to 0. A quadratic algebra $\mathbf{A}$ is said to be Koszul if the Tor groups $\mathrm{Tor}^{\mathbf{A}}_{n,(m)}(\mathbb{K},\mathbb{K})$ (n is the homological degree and m is graduation induced by the natural graduation over $\mathbf{A}$) vanish for $m \neq n$.

A property of Koszul algebras is that the ground field $\mathbb{K}$ admits a *Koszul resolution*. The name of this resolution is due to the fact that it is inspired by ideas of Koszul (see [Kos50]). Let $\mathbf{A}$ be a quadratic algebra and let $\langle X \mid R\rangle$ be a quadratic presentation of $\mathbf{A}$. We denote by $\mathbb{K}X$ and $\overline{R}$ the vector space spanned by X and the sub-vector space of $\mathbb{K}X^{\otimes 2}$ spanned by R, respectively. The *Koszul complex* of a $\mathbf{A}$ is the complex of free left $\mathbf{A}$-modules:

$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{A}\otimes J_n \xrightarrow{\partial_n} \mathbf{A}\otimes J_{n-1} \longrightarrow \cdots \xrightarrow{\partial_4} \mathbf{A}\otimes J_3 \xrightarrow{\partial_3} \mathbf{A}\otimes \overline{R} \xrightarrow{\partial_2} \mathbf{A}\otimes \mathbb{K}X \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0,$$

where, for every integer n such that $n \geq 2$, we have:

$$J_n = \bigcap_{i=0}^{n-2} \mathbb{K}X^{\otimes i} \otimes \overline{R} \otimes \mathbb{K}X^{\otimes n-2-i}.$$

The differentials of the Koszul complex are defined by the inclusions of $\overline{R}$ in $\mathbf{A}\otimes\mathbb{K}X$, of J_3 in $\mathbf{A}\otimes\overline{R}$ and of J_n in $\mathbf{A}\otimes J_{n-1}$ for every integer n such that $n \geq 4$. Then, a quadratic algebra is Koszul if and only if its Koszul complex is acyclic, that is, if and only if the Koszul complex of $\mathbf{A}$ is a resolution of $\mathbb{K}$.

Another characterisation of Koszulness was given by Backelin in [BF85] (see also Theorem 4.1 in [PP05, chapter 2]): a quadratic algebra is Koszul if and only if it is *distributive* (that means that some lattices defined with X and R are distributive). Moreover, Koszul algebras have been studied through computational approaches based on a monomial order, that is, a well founded total order on the set of monomials. In [Ani86], Anick used Gröbner basis to construct a free resolution of $\mathbb{K}$ (see also [Ufn95, Section 3.8]). This resolution enables us to conclude that an algebra which admits a quadratic Gröbner basis is Koszul. In [Ber98], Berger studied quadratic algebras with a *side-confluent presentation* [1]. The latter is a transcription of the notion of quadratic Gröbner basis using some linear operators. More precisely, we can associate with any quadratic presentation $\langle X \mid R\rangle$ of $\mathbf{A}$ a unique linear projector S of $\mathbb{K}X^{\otimes 2}$. This projector maps any element of $\mathbb{K}X^{\otimes 2}$ to a better one with respect to the monomial order. The presentation $\langle X \mid R\rangle$ is said to be side-confluent if there exists an integer k such that:

$$\langle S\otimes \mathrm{Id}_{\mathbb{K}X}, \mathrm{Id}_{\mathbb{K}X}\otimes S\rangle^k = \langle \mathrm{Id}_{\mathbb{K}X}\otimes S, S\otimes \mathrm{Id}_{\mathbb{K}X}\rangle^k,$$

where $\langle t,s\rangle^k$ denotes the product $\cdots sts$ with k factors. The algebra $\mathscr{A}_k$ presented by:

$$\left\langle s_1, s_2 \mid \langle s_1,s_2\rangle^k = \langle s_2,s_1\rangle^k,\ s_i^2 = s_i,\ i=1,2 \right\rangle,$$

is naturally associated with a side-confluent presentation. This algebra is the *confluence algebra of degree* k. In [Ber98, Section 5], Berger used specific representations of these algebras to construct a contracting homotopy for the Koszul complex of a quadratic algebra admitting a side-confluent presentation. This construction enables us to conclude that such an algebra is Koszul.

N-Koszul algebras. Let N be an integer such that $N \geq 2$. An *N-homogeneous algebra* is a graded associative algebra over a field $\mathbb{K}$ which admits an *N-homogeneous presentation* $\langle X \mid R\rangle$, that is, X is a set of generators and R is a set of N-homogeneous relations. In [Ber01] the notion of Koszul algebra

[1] This notion corresponds to the one of *X-confluent algebra* in [Ber98]. However, we prefer to use our terminology because the property of confluence depends on the presentation.

is extended to the notion of *N-Koszul algebra.* An N-homogeneous algebra $\mathbf{A}$ is said to be N-Koszul if the Tor groups $\mathrm{Tor}^{\mathbf{A}}_{n,(m)}(\mathbb{K},\mathbb{K})$ vanish for $m \neq l_N(n)$, where l_N is the function defined by:

$$l_N(n) = \begin{cases} kN, & \text{if } n = 2k, \\ kN+1, & \text{if } n = 2k+1. \end{cases}$$

We remark that a 2-Koszul algebra is precisely a Koszul algebra. Thus, the notion of N-Koszul algebra generalises the one of Koszul algebra.

In the same paper, Berger defined the *Koszul complex* of an N-homogeneous algebra. Let $\langle X \mid R\rangle$ be an N-homogeneous presentation of $\mathbf{A}$. The Koszul complex of $\mathbf{A}$ is the complex of left $\mathbf{A}$-modules:

$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{A}\otimes J_n^N \xrightarrow{\partial_n} \mathbf{A}\otimes J_{n-1}^N \longrightarrow \cdots \xrightarrow{\partial_4} \mathbf{A}\otimes J_3^N \xrightarrow{\partial_3} \mathbf{A}\otimes \overline{R} \xrightarrow{\partial_2} \mathbf{A}\otimes \mathbb{K}X \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0,$$

where the vector spaces J_n^N are defined by:

$$J_n^N = \bigcap_{i=0}^{l_N(n)-N} \mathbb{K}X^{\otimes i}\otimes \overline{R}\otimes \mathbb{K}X^{\otimes l_N(n)-N-i}.$$

As in the quadratic case, this complex characterises the property of N-Koszulness: an N-homogeneous algebra is N-Koszul if and only if its Koszul complex is acyclic (see [Ber01, Proposition 2.12]). This complex also find applications in the study of higher Koszul duality (see [DV13]).

Berger studied the property of N-Koszulness using monomial orders. As in the quadratic case, there exists a unique linear projector S of $\mathbb{K}X^{\otimes N}$ associated with an N-homogeneous presentation of $\mathbf{A}$ which maps any element to a better one with respect to the monomial order. Then, a presentation is *side-confluent* if for every integer m such that $N+1 \leq m \leq 2N-1$, there exists an integer k which satisfies:

$$\langle S\otimes \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}, \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}\otimes S\rangle^k = \langle \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}\otimes S, S\otimes \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}\rangle^k .$$

Contrary to the quadratic case, an algebra admitting a side-confluent presentation is not necessarily N-Koszul. Indeed, when the set X is finite, such an algebra is N-Koszul if and only if the *extra-condition* holds (see [Ber01, Proposition 3.4]). The extra-condition is stated as follows:

$$(ec): \quad \left(\mathbb{K}X^{\otimes m}\otimes \overline{R}\right)\cap\left(\overline{R}\otimes \mathbb{K}X^{\otimes m}\right) \subset \mathbb{K}X^{\otimes m-1}\otimes \overline{R}\otimes \mathbb{K}X, \text{ for every } 2\leq m\leq N-1.$$

We group these hypothesis in the following definition:

2.3.2 Definition. Let $\mathbf{A}$ be an N-homogeneous algebra. A side-confluent presentation $\langle X \mid R\rangle$ such that X is finite and the extra-condition holds is said to be *extra-confluent.*

Our problematic. We deduce of the works from [Ber01] that the Koszul complex of an algebra $\mathbf{A}$ admitting an extra-confluent presentation is acyclic. However, there does not exist an explicit contracting homotopy for the Koszul complex of $\mathbf{A}$. The purpose of this paper is to construct such a contracting homotopy. For the quadratic case, our contracting homotopy is the one constructed in [Ber98, Section 5].

Our results

We present the different steps of our construction. Recall that an extra-confluent presentation needs a monomial order. Thus, in what follows, we fix a monomial order. For every integer m, we denote by $X^{(m)}$ the set of words of length m.

Reduction pairs associated with a presentation. In the way to construct our contracting homotopy, we will associate with any N-homogeneous presentation $\langle X \mid R\rangle$ such that X is finite, a family $P_{n,m} = (F_1^{n,m}, F_2^{n,m})$, where $F_1^{n,m}$ and $F_2^{n,m}$ are linear projectors of $\mathbb{K}X^{(m)}$. The pair $P_{n,m}$ is called the *reduction pair of bi-degree* (n,m) associated with $\langle X \mid R\rangle$. We point the fact that the finiteness condition over X will be necessary to define the operators $F_i^{n,m}$. Moreover, these operators satisfy the following condition: for any $w \in X^{(m)}$, $F_i^{n,m}(w)$ is either equal to w or is a sum of monomials which are strictly smaller than w with respect to the monomial order. The linear projectors of $\mathbb{K}X^{(m)}$ satisfying the previous condition are called *reduction operators relatively to* $X^{(m)}$. The set of reduction operators relatively to $X^{(m)}$ admits a lattice structure (we will recall it in Section 3.2). This structure plays an essential role in our constructions. A pair (T_1, T_2) of reduction operators relatively to $X^{(m)}$ is said to be *confluent* if there exists an integer k such that we have the following equality in $\operatorname{End}\left(\mathbb{K}X^{(m)}\right)$:

$$\langle T_1, T_2\rangle^k = \langle T_2, T_1\rangle^k .$$

Then, our first result is:

4.1.4 Theorem. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra admitting a side-confluent presentation* $\langle X \mid R\rangle$*, where* X *is a finite set. The reduction pairs associated with* $\langle X \mid R\rangle$ *are confluent.*

The left bound of a side-confluent presentation. The reduction pairs associated with a side-confluent presentation $\langle X \mid R\rangle$ enable us to define a family of representations of confluence algebras in the following way:

$$\begin{aligned} \varphi^{P_{n,m}} : \left\langle s_1, s_2 \mid \langle s_1, s_2\rangle^{k_{n,m}} = \langle s_2, s_1\rangle^{k_{n,m}},\ s_i^2 = s_i,\ i = 1,2 \right\rangle &\longrightarrow \operatorname{End}\left(\mathbb{K}X^{(m)}\right), \\ s_i &\longmapsto F_i^{n,m} \end{aligned}$$

where the integer $k_{n,m}$ satisfies:

$$\langle F_1^{n,m}, F_2^{n,m}\rangle^{k_{n,m}} = \langle F_2^{n,m}, F_1^{n,m}\rangle^{k_{n,m}} .$$

For every integers n and m we will consider a specific element in $\mathscr{A}_{k_{n,m}}$:

$$\gamma_1 = (1 - s_2)\left(s_1 + s_1 s_2 s_1 + \cdots + \langle s_2, s_1\rangle^{2i+1}\right),$$

where the integer i depends on $k_{n,m}$. The shape of this element will be motivated in Section 3.1. In Section 4.2 we will use the elements $\varphi^{P_{n,m}}(\gamma_1)$ to construct a family of $\mathbb{K}$-linear maps

$$\begin{aligned} &h_0 : \mathbf{A} \longrightarrow \mathbf{A} \otimes \mathbb{K}X, \\ &h_1 : \mathbf{A} \otimes \mathbb{K}X \longrightarrow \mathbf{A} \otimes \overline{R}, \\ &h_2 : \mathbf{A} \otimes \overline{R} \longrightarrow \mathbf{A} \otimes J_3^N, \\ &h_n : \mathbf{A} \otimes J_n^N \longrightarrow \mathbf{A} \otimes J_{n+1}^N, \text{ for } n \geq 3, \end{aligned}$$

where $\mathbf{A}$ is the N-homogeneous algebra presented by $\langle X \mid R\rangle$. The family $(h_n)_n$ is called the *left bound of* $\langle X \mid R\rangle$. In Proposition 4.2.5, we will show that the left bound of $\langle X \mid R\rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$ if and only if $\langle X \mid R\rangle$ satisfies some identities. These identities are called the *reduction relations*.

Extra-confluent presentations and reduction relations. Finally, we will show that the extra-condition implies that the reduction relations hold. Then, our main result is stated as follows:

4.3.5 Theorem. *Let $\mathbf{A}$ be an N-homogeneous algebra admitting an extra-confluent presentation $\langle X \mid R\rangle$. The left bound of $\langle X \mid R\rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$.*

Organisation

In Section 2, we recall how we can construct the Koszul complex of an N-homogeneous algebra. We also recall the definition of an extra-confluent presentation. In Section 3.1, we make explicit our construction in small homological degree. In Section 3.2, we recall the definitions of confluence algebras and reduction operators. We also recall the link between reduction operators and representations of confluence algebras. In Section 4, we construct the contracting homotopy in terms of confluence. As an illustration of our construction, we provide in Section 5 three examples: the *symmetric algebra*, *monomial algebras* which satisfy the *overlap properties* and the *enveloping algebra of the Heisenberg Lie algebra.*

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2 Preliminaries

2.1 The Koszul complex

2.1.1. Conventions and notations. We denote by $\mathbb{K}$ a field. We say vector space and algebra instead of $\mathbb{K}$-vector space and $\mathbb{K}$-algebra, respectively. We consider only associative algebras. Given a set X, we denote by $\langle X\rangle$ and $\mathbb{K}X$ the free monoid and the vector space spanned by X, respectively. For every integer m, we denote by $X^{(m)}$ the subset of $\langle X\rangle$ of words of length m.

We write $V = \mathbb{K}X$. We identify $\mathbb{K}X^{(m)}$ and the free algebra $\mathbb{K}\langle X\rangle$ spanned by X to $V^{\otimes m}$ and to the tensor algebra $\mathrm{T}(V)$ over V, respectively.

Let $\mathbf{A}$ be an algebra. A *presentation* of $\mathbf{A}$ is a pair $\langle X \mid R\rangle$, where X is a set and R is a subset of $\mathbb{K}\langle X\rangle$ such that $\mathbf{A}$ is isomorphic to the quotient of $\mathbb{K}\langle X\rangle$ by the two-sided ideal spanned by R. The latter is denoted by $I(R)$, and the isomorphism from $\mathbf{A}$ to $\mathbb{K}\langle X\rangle/I(R)$ is denoted by $\psi_{\langle X\mid R\rangle}$. For every $f \in \mathbb{K}\langle X\rangle$, we denote by $\overline{f}$ the image of f through the natural projection of $\mathbb{K}\langle X\rangle$ over $\mathbf{A}$.

Let N be an integer such that $N \geq 2$. An *N-homogeneous presentation* of $\mathbf{A}$ is a presentation $\langle X \mid R\rangle$ of $\mathbf{A}$ such that R is included in $V^{\otimes N}$. In this case, the two-sided ideal $I(R)$ is the direct sum of vector spaces $I(R)_m$ defined by $I(R)_m = 0$ if $m < N$, and

$$I(R)_m = \sum_{i=0}^{m-N} V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-N-i} \text{ if } m \geq N,$$

where $\overline{R}$ denotes the sub-vector space of $V^{\otimes N}$ spanned by R. An *N-homogeneous algebra* is a graded algebra $\mathbf{A} = \bigoplus_{m\in\mathbb{N}} \mathbf{A}_m$ which admits an N-homogeneous presentation $\langle X \mid R\rangle$ such that for every integer m, $\psi_{\langle X\mid R\rangle}$ induces a $\mathbb{K}$-linear isomorphism from $\mathbf{A}_m$ to $V^{\otimes m}/I(R)_m$:

$$\begin{aligned}\mathbf{A} &= \bigoplus_{m\in\mathbb{N}} \mathbf{A}_m \\ &\simeq \mathbb{K} \oplus V \oplus \cdots \oplus V^{\otimes N-1} \oplus \frac{V^{\otimes N}}{\overline{R}} \oplus \frac{V^{\otimes N+1}}{V\otimes\overline{R} + \overline{R}\otimes V} \oplus \cdots\end{aligned}$$

We denote by $\varepsilon : \mathbf{A} \longrightarrow \mathbb{K}$ the projection which maps $1_{\mathbf{A}}$ to $1_{\mathbb{K}}$ and $\mathbf{A}_m$ to 0 for every integer m such that $m \geq 1$.

2.1.2. The construction of the Koszul complex. Let $\mathbf{A}$ be an N-homogeneous algebra and let $\langle X \mid R\rangle$ be an N-homogeneous presentation of $\mathbf{A}$. We write $V = \mathbb{K}X$. We consider the family of vector spaces $(J_n^N)_n$ defined by $J_0^N = \mathbb{K}$, $J_1^N = V$, $J_2^N = \overline{R}$ and for every integer n such that $n \geq 3$

$$J_n^N = \bigcap_{i=0}^{l_N(n)-N} V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes l_N(n)-N-i},$$

where the function $l_N : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by

$$l_N(n) = \begin{cases} kN, & \text{if } n = 2k, \\ kN+1, & \text{if } n = 2k+1. \end{cases}$$

When there is no ambiguity, we write J_n instead of J_n^N.

Let n be an integer. For every $w \in X^{(l_N(n+1))}$, let $w_1 \in X^{(l_N(n+1)-l_N(n))}$ and $w_2 \in X^{(l_N(n))}$ such that $w = w_1 w_2$. Let us consider the $\mathbf{A}$-linear map

$$\begin{aligned} F_{n+1} \colon \mathbf{A} \otimes V^{\otimes l_N(n+1)} &\longrightarrow \mathbf{A} \otimes V^{\otimes l_N(n)}. \\ 1_{\mathbf{A}} \otimes w &\longmapsto \overline{w_1} \otimes w_2 \end{aligned}$$

Recall from [Ber01, Section 3] that the *Koszul complex* of $\mathbf{A}$ is the complex $(K_\bullet, \partial)$

$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{A} \otimes J_n \xrightarrow{\partial_n} \mathbf{A} \otimes J_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} \mathbf{A} \otimes J_1 \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0,$$

where ∂_n is the restriction of F_n to $\mathbf{A} \otimes J_n$. In particular, the map ∂_1 is defined by $\partial_1(1_{\mathbf{A}} \otimes x) = \overline{x}$ for every $x \in X$.

2.1.3. Remark. The two following remarks show that the Koszul complex is well-defined:

1. Let n be an integer. The vector space J_{n+1} is included in $V^{\otimes l_N(n+1)-l_N(n)} \otimes J_n$. Thus, the vector space $F_{n+1}(\mathbf{A} \otimes J_{n+1})$ is included in $\mathbf{A} \otimes J_n$.
2. Let n be an integer such that $n \geq 1$. The vector space J_{n+1} is included in $R \otimes J_{n-1}$. Thus, the restriction of $F_n F_{n+1}$ to $\mathbf{A} \otimes J_{n+1}$ vanishes.

2.1.4. Example. We consider the enveloping algebra of the Heisenberg Lie algebra introduced in [AS87]. This is the 3-homogeneous algebra presented by

$$\langle x_1, x_2 \mid x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2,\ x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2\rangle.$$

This algebra is the minimal (with respect to the number of generators) example of *Yang-Mills algebra* introduced in [CDV02].

The map $\partial_2 : \mathbf{A} \otimes \overline{R} \longrightarrow \mathbf{A} \otimes V$ is defined by

$$\begin{aligned} &\partial_2(1_{\mathbf{A}} \otimes x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2) = \overline{x_2x_1} \otimes x_1 - 2\overline{x_1x_2} \otimes x_1 + \overline{x_1x_1} \otimes x_2, \text{ and} \\ &\partial_2(1_{\mathbf{A}} \otimes x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2) = \overline{x_2x_2} \otimes x_1 - 2\overline{x_2x_1} \otimes x_2 + \overline{x_1x_2} \otimes x_2. \end{aligned}$$

The vector space $J_3 = \left(V \otimes \overline{R}\right) \cap \left(\overline{R} \otimes V\right)$ is the one-dimensional vector space spanned by

$$\begin{aligned} v &= x_2(x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2) + x_1(x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2) \\ &= (x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2)x_1 + (x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2)x_2. \end{aligned}$$

The map $\partial_3 : \mathbf{A} \otimes J_3 \longrightarrow \mathbf{A} \otimes \overline{R}$ is defined by

$$\partial_3(1_{\mathbf{A}} \otimes v) = \overline{x_2} \otimes (x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2) + \overline{x_1} \otimes (x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2).$$

2.2 Side-confluent presentations

Through this section we fix an N-homogeneous algebra $\mathbf{A}$ and an N-homogeneous presentation $\langle X \mid R\rangle$ of $\mathbf{A}$. We assume that X is a totally ordered set. We write $V = \mathbb{K}X$.

2.2.1. Reductions. For every integer m, the set $X^{(m)}$ is totally ordered for the lexicographic order induced by the order over X. For every $f \in V^{\otimes m} \setminus \{0\}$, the *leading monomial of* f, denoted by $\mathrm{lm}\,(f)$, is the greatest element of $X^{(m)}$ occurring in the decomposition of f. We denote by $\mathrm{lc}(f)$ the coefficient of $\mathrm{lm}\,(f)$ in the decomposition of f. Let

$$R' = \left\{ \frac{1}{\mathrm{lc}\,(f)} f,\ f \in R \right\}.$$

Then, $\langle X \mid R'\rangle$ is an N-homogeneous presentation of $\mathbf{A}$. Thus, we can assume that $\mathrm{lc}\,(f)$ is equal to 1 for every $f \in R$.

For every $w_1, w_2 \in \langle X\rangle$ and every $f \in R$, let $r_{w_1 f w_2}$ be the $\mathbb{K}$-linear endomorphism of $\mathrm{T}\,(V)$ defined on the basis $\langle X\rangle$ in the following way:

$$r_{w_1 f w_2}\,(w) = \begin{cases} w_1\,(\mathrm{lm}\,(f) - f)\,w_2, & \text{if } w = w_1 \mathrm{lm}\,(f)\,w_2, \\ w, & \text{otherwise.} \end{cases}$$

Taking the terminology of [Ber78], the morphisms $r_{w_1 f w_2}$ are called the *reductions of* $\langle X \mid R\rangle$.

2.2.2. Normal forms. An element $f \in \mathrm{T}\,(V)$ is said to be a *normal form for* $\langle X \mid R\rangle$ if $r(f) = f$ for every reduction r of $\langle X \mid R\rangle$. Given an element f of $\mathrm{T}\,(V)$, a *normal form of* f is a normal form g such that there exist reductions $r_1, \cdots, r_n$ satisfying $g = r_1 \cdots r_n(f)$. In this case, we have $\overline{f} = \overline{g}$.

The presentation $\langle X \mid R\rangle$ is said to be *reduced* if, for every $f \in R$, $\mathrm{lm}\,(f) - f$ is a normal form for $\langle X \mid R\rangle$ and $\mathrm{lm}\,(f)$ is a normal form for $\langle X \mid R \setminus \{f\}\rangle$. From this moment, all the presentations are assumed to be reduced.

2.2.3. Critical branching. A *critical branching of* $\langle X \mid R\rangle$ is a 5-tuple (w_1, w_2, w_3, f, g) where $f,\ g \in R$ and $w_1,\ w_2,\ w_3$ are non empty words such that:

$$w_1 w_2 = \mathrm{lm}\,(f)\,, \text{ and}$$
$$w_2 w_3 = \mathrm{lm}\,(g)\,.$$

The word $w_1 w_2 w_3$ is the *source* of this critical branching.

2.2.4. The operator of a presentation. Let S be the endomorphism of $V^{\otimes N}$ defined on the basis $X^{(N)}$ in the following way:

$$S\,(w) = \begin{cases} \mathrm{lm}\,(f) - f, & \text{if there exists } f \in R \text{ such that } w = \mathrm{lm}\,(f)\,, \\ w, & \text{otherwise.} \end{cases}$$

The operator S is *the operator of* $\langle X \mid R\rangle$. The presentation $\langle X \mid R\rangle$ is reduced. Thus, S is well-defined and is a projector. The kernel of S is equal to $\overline{R}$. If $w \in X^{(N)}$ is a normal form, then $S(w)$ is equal to w. If w is not a normal form, then $S(w)$ is a linear combination of words strictly smaller than w.

2.2.5. Definition. The presentation $\langle X \mid R\rangle$ is said to be *side-confluent* if for every integer m such that $1 \leq m \leq N-1$, there exists an integer k such that:

$$\left\langle \mathrm{Id}_{V^{\otimes m}} \otimes S, S \otimes \mathrm{Id}_{V^{\otimes m}} \right\rangle^k = \left\langle S \otimes \mathrm{Id}_{V^{\otimes m}}, \mathrm{Id}_{V^{\otimes m}} \otimes S \right\rangle^k,$$

where $\langle t, s\rangle^k$ denotes the product $\cdots sts$ with k factors.

The Diamond Lemma ([Ber78, Theorem 1.2]) implies the following:

2.2.6. Proposition. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra. Assume that* $\mathbf{A}$ *admits a side-confluent presentation* $\langle X \mid R\rangle$*.* Then, the following hold:

1. *Every element of* $\mathrm{T}(V)$ *admits a unique normal form for* $\langle X \mid R\rangle$*.*
2. *The set* $\{\overline{w},\ w \in \langle X\rangle$ *is a normal form*$\}$ *is a basis of* $\mathbf{A}$.
3. *An element of* $\mathrm{T}(V)$ *belongs to* $I(R)$ *if and only if its normal form is equal to* 0.

Proof. Let S be the operator of $\langle X \mid R\rangle$. Let (w_1, w_2, w_3, f, g) be a critical branching of $\langle X \mid R\rangle$. Let m be the length of $w = w_1w_2w_3$. The presentation $\langle X \mid R\rangle$ being N-homogeneous, we have $N+1 \leq m \leq 2N-1$. Thus, there exists an integer k such that:

$$\langle \mathrm{Id}_{V^{\otimes m-N}} \otimes S, S \otimes \mathrm{Id}_{V^{\otimes m-N}}\rangle^k (w) = \langle S \otimes \mathrm{Id}_{V^{\otimes m-N}}, \mathrm{Id}_{V^{\otimes m-N}} \otimes S\rangle^k (w).$$

Hence, there exist two sequences of reductions $r_1, \cdots, r_n$ and $r'_1, \cdots, r'_l$ such that $r_1 \cdots r_n\left((\mathrm{lm}\,(f) - f)\, w_3\right)$ is equal to $r'_1 \cdots r'_l\left(w_1\,(\mathrm{lm}\,(g) - g)\right)$. We deduce from [Ber78, Theorem 1.2] that every element $f \in \mathrm{T}(V)$ admits a unique normal form for $\langle X \mid R\rangle$ and that $\{\overline{w},\ w \in \langle X\rangle$ is a normal form$\}$ is a basis of $\mathbf{A}$. Thus, the two first points hold.

Let us show the third point. Let f be an element of $\mathrm{T}(V)$ and let $\widehat{f}$ be its unique normal form. We write:

$$\widehat{f} = \sum_{i\in I} \lambda_i w_i,$$

where $w_i \in \langle X\rangle$ are normal forms. Then, $\overline{f}$ is equal to $\sum_{i\in I} \lambda_i \overline{w_i}$. From the second point, $\overline{f}$ is equal to 0 if and only if λ_i is equal to 0 for every $i \in I$.

□

2.2.7. Lemma. *Assume that the presentation* $\langle X \mid R\rangle$ *is side-confluent. Let* S *be the operator of* $\langle X \mid R\rangle$*. For every integer* m *such that* $N+1 \leq m \leq 2N-1$*, there exists an integer* k *such that:*

$$\begin{aligned}&\langle \mathrm{Id}_{V^{\otimes m}} - \mathrm{Id}_{V^{\otimes m-N}} \otimes S, \mathrm{Id}_{V^{\otimes m}} - S \otimes \mathrm{Id}_{V^{\otimes m-N}}\rangle^k \\ =&\langle \mathrm{Id}_{V^{\otimes m}} - S \otimes \mathrm{Id}_{V^{\otimes m-N}}, \mathrm{Id}_{V^{\otimes m}} - \mathrm{Id}_{V^{\otimes m-N}} \otimes S\rangle^k .\end{aligned}$$

Moreover, for every $w \in X^{(m)}$ *such that* $\mathrm{Id}_{V^{\otimes m-N}} \otimes S(w)$ *and* $S \otimes \mathrm{Id}_{V^{\otimes m-N}}(w)$ *are different from* w*, we have*:

$$\mathrm{lm}\left((\langle \mathrm{Id}_{V^{\otimes m}} - \mathrm{Id}_{V^{\otimes m-N}}\rangle \otimes S, \mathrm{Id}_{V^{\otimes m}} - S \otimes \mathrm{Id}_{V^{\otimes m-N}})^k (w)\right) = w.$$

Proof. We write $S_1 = \mathrm{Id}_{V^{\otimes m-N}} \otimes S$ and $S_2 = S \otimes \mathrm{Id}_{V^{\otimes m-N}}$.

The presentation $\langle X \mid R\rangle$ is side-confluent. Thus, there exists $k \in \mathbb{N}$ such that $\langle S_2, S_1\rangle^k$ is equal to $\langle S_1, S_2\rangle^k$. The morphisms S_1 and S_2 being projectors, we show by induction that for every integer j we have:

$$\langle \mathrm{Id}_{V^{\otimes m}} - S_1, \mathrm{Id}_{V^{\otimes m}} - S_2\rangle^j = \mathrm{Id}_{V^{\otimes m}} + \sum_{i=1}^{j-1} (-1)^i \left(\langle S_1, S_2\rangle^i + \langle S_2, S_1\rangle^i\right) + (-1)^j \langle S_1, S_2\rangle^j ,$$

$$\langle \mathrm{Id}_{V^{\otimes m}} - S_2, \mathrm{Id}_{V^{\otimes m}} - S_1\rangle^j = \mathrm{Id}_{V^{\otimes m}} + \sum_{i=1}^{j-1} (-1)^i \left(\langle S_1, S_2\rangle^i + \langle S_2, S_1\rangle^i\right) + (-1)^j \langle S_2, S_1\rangle^j .$$

In particular we have:

$$\langle \mathrm{Id}_{V^{\otimes m}} - S_2, \mathrm{Id}_{V^{\otimes m}} - S_1\rangle^k = \langle \mathrm{Id}_{V^{\otimes m}} - S_1, \mathrm{Id}_{V^{\otimes m}} - S_2\rangle^k .$$

Moreover, if $w \in X^{(m)}$ is such that $S_1(w)$ and $S_2(w)$ are different from w, then $S_1(w)$ and $S_2(w)$ are strictly smaller than w. We deduce from the relation

$$\langle \mathrm{Id}_{V^{\otimes m}} - S_1, \mathrm{Id}_{V^{\otimes m}} - S_2 \rangle^k (w) = w + \sum_{i=1}^{k-1} (-1)^i \left(\langle S_1, S_2 \rangle^i + \langle S_2, S_1 \rangle^i \right)(w) + (-1)^k \langle S_1, S_2 \rangle^k (w),$$

that $\mathrm{lm}\left(\langle \mathrm{Id}_{V^{\otimes m}} - S_1, \mathrm{Id}_{V^{\otimes m}} - S_2 \rangle^k (w) \right)$ is equal to w. □

2.2.8. Example. We consider the presentation from Example 2.1.4 of the enveloping algebra of the Heisenberg Lie algebra with the order $x_1 < x_2$. It was proven in [KVdB15, Theorem 6.3.2] that this presentation is side-confluent (in fact, B.Kriegk and M.Van den Bergh have proven that any Yang-Mills algebra admits a side-confluent presentation). We propose there an other proof of this result.

The operator $S \in \mathrm{End}\left(V^{\otimes 3}\right)$ of this presentation is defined on the basis $X^{(3)}$ by

$$S(w) = \begin{cases} 2x_1x_2x_1 - x_1x_1x_2, & \text{if } w = x_2x_1x_1, \\ 2x_2x_1x_2 - x_1x_2x_2, & \text{if } w = x_2x_2x_1, \\ w, & \text{otherwise.} \end{cases}$$

This presentation admits exactly one critical branching:

$$(x_2, x_2x_1, x_1, x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2,\ x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2).$$

We have:

$$\begin{aligned} \langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S \rangle^2 (x_2x_2x_1x_1) &= \langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V \rangle^2 (x_2x_2x_1x_1) \\ &= x_2x_1x_2x_1 - 2x_1x_2x_1x_2 + x_1x_1x_2x_2. \end{aligned}$$

Moreover, for every $w \in X^{(4)}$ which is different from $x_2x_2x_1x_1$, we check that $\langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S \rangle^2 (w)$ is equal to $\langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V \rangle^2 (w)$. Thus, we have:

$$\langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S \rangle^2 = \langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V \rangle^2.$$

For every $w \in X^{(5)}$ we check that $\langle S \otimes \mathrm{Id}_{V^{\otimes 2}}, \mathrm{Id}_{V^{\otimes 2}} \otimes S \rangle^2 (w)$ and $\langle \mathrm{Id}_{V^{\otimes 2}} \otimes S, S \otimes \mathrm{Id}_{V^{\otimes 2}} \rangle^2 (w)$ are equal. Thus, we have:

$$\langle S \otimes \mathrm{Id}_{V^{\otimes 2}}, \mathrm{Id}_{V^{\otimes 2}} \otimes S \rangle^2 = \langle \mathrm{Id}_{V^{\otimes 2}} \otimes S, S \otimes \mathrm{Id}_{V^{\otimes 2}} \rangle^2.$$

We conclude that the presentation from Example 2.1.4 with the order $x_1 < x_2$ is side-confluent.

2.3 Extra-confluent presentations

2.3.1. The extra-condition. Let $\mathbf{A}$ be an N-homogeneous algebra. Assume that $\mathbf{A}$ admits a side-confluent presentation $\langle X \mid R \rangle$ where X is a totally ordered finite set. Recall from [Ber01, Section 3] that the Koszul complex of $\mathbf{A}$ is acyclic if and only if the *extra-condition* holds. The extra-condition is stated as follows:

$$\left(\mathbb{K}X^{(n)} \otimes \overline{R}\right) \cap \left(\overline{R} \otimes \mathbb{K}X^{(n)}\right) \ \subset\ \mathbb{K}X^{(n-1)} \otimes \overline{R} \otimes \mathbb{K}X, \text{ for every } 2 \leq n \leq N-1.$$

2.3.2. Definition. Let $\mathbf{A}$ be an N-homogeneous algebra. A side-confluent presentation $\langle X \mid R \rangle$ such that X is finite and the extra-condition holds is said to be *extra-confluent.*

2.3.3. Remark. If $N = 2$, the extra-condition is an empty condition. Thus, in this case, the notions of extra-confluent presentation and side-confluent presentation coincide.

An extra-confluent presentation has the following interpretation in terms of critical branching:

2.3.4. Proposition. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra. Assume that* $\mathbf{A}$ *admits an extra-confluent presentation* $\langle X \mid R\rangle$*. Let* $w = x_1 \cdots x_m$ *be the source of a critical branching of* $\langle X \mid R\rangle$*. The word* $x_{m-N} \cdots x_{m-1}$ *is not a normal form for* $\langle X \mid R\rangle$*.*

Proof. The presentation $\langle X \mid R\rangle$ is N-homogeneous. In particular, we have $N+1 \leq m \leq 2N-1$. If $m = N+1$, there is nothing to prove. Thus, we assume that m is greater than $N+2$.

Let S be the operator of $\langle X \mid R\rangle$. We write

$$S_1 = S \otimes \mathrm{Id}_{\otimes m-N} \text{ and } S_2 = \mathrm{Id}_{\otimes m-N} \otimes S.$$

The presentation $\langle X \mid R\rangle$ is side-confluent. Thus, from Lemma 2.2.7, there exists an integer k such that

$$\langle \mathrm{Id}_{V^{\otimes m}} - S_2, \mathrm{Id}_{V^{\otimes m}} - S_1\rangle^k = \langle \mathrm{Id}_{V^{\otimes m}} - S_1, \mathrm{Id}_{V^{\otimes m}} - S_2\rangle^k .$$

We denote by Λ this common morphism. By hypothesis, $S_1(w)$ and $S_2(w)$ are different from w. From Lemma 2.2.7, $\mathrm{lm}\,(\Lambda(w))$ is equal to w.

The image of Λ is included in $\mathrm{im}\,(\mathrm{Id}_{\mathbb{K}X^{(m)}} - S_1) \cap \mathrm{im}\,(\mathrm{Id}_{\mathbb{K}X^{(m)}} - S_2)$ that is, $\ker(S_1) \cap \ker(S_2)$. The latter is equal to $\overline{R} \otimes \mathbb{K}X^{(m-N)} \cap \mathbb{K}X^{(m-N)} \otimes \overline{R}$. The presentation $\langle X \mid R\rangle$ satisfies the extra-condition. Thus, the image of Λ is included in $\mathbb{K}X^{(m-N-1)} \otimes \overline{R} \otimes \mathbb{K}X$. In particular, there exist $w_1, \cdots, w_l \in X^{(m-N-1)}$, $f_1, \cdots, f_l \in R$, $x_1, \cdots, x_l \in X$ and $\lambda_1, \cdots \lambda_l \in \mathbb{K}$ which satisfy

$$\Lambda(w) = \sum_{i=1}^{l} \lambda_i w_i f_i x_i.$$

Thus, $\mathrm{lm}\,(\Lambda(w)) = w$ is equal to $w_i \mathrm{lm}\,(f_i)\, x_i$ for some $1 \leq i \leq l$. We conclude that $x_{n-N} \cdots x_{m-1}$ is equal to $\mathrm{lm}\,(f_i)$. In particular, it is not a normal form. □

2.3.5. Remark. Let $\mathbf{A}$ be the algebra presented by $\langle x < y \mid xyx\rangle$. This presentation is side-confluent. There is only one critical branching: (xy, x, yx, xyx, xyx). The source $xyxyx$ of this critical has length 5. We deduce from Proposition 2.3.4 that the extra-condition does not hold.

Let us check that the Koszul complex of $\mathbf{A}$ is not acyclic: the vector space J_3 is reduced to $\{0\}$ and the map $\partial_2 : \mathbf{A} \otimes \overline{R} \longrightarrow \mathbf{A} \otimes V$ is defined by $\partial_2\,(1_{\mathbf{A}} \otimes xyx) = \overline{xy} \otimes x$. In particular, $\overline{xy} \otimes xyx$ belongs to the kernel of ∂_2. Thus, we have a strict inclusion $\mathrm{im}\,(\partial_3) \subsetneq \ker(\partial_2)$.

2.3.6. Example. We consider the presentation from Example 2.2.8. The vector space $V^{\otimes 2} \otimes \overline{R} \cap \overline{R} \otimes V^{\otimes 2}$ is reduced to $\{0\}$. Then, the extra-condition holds. We conclude that the presentation from Example 2.2.8 is extra-confluent.

3 Confluence algebras and reduction operators

3.1 The contracting homotopy in small degree

Through this section we fix an N-homogeneous algebra $\mathbf{A}$. We assume that $\mathbf{A}$ admits an extra-confluent presentation $\langle X \mid R\rangle$. This presentation is also fixed. We write $V = \mathbb{K}X$.

The aim of this section is to make explicit our contracting homotopy in small homological degree. The formal construction will be done in Section 4.

We have to construct a family of $\mathbb{K}$-linear maps

$$h_{-1} : \mathbb{K} \longrightarrow \mathbf{A}, \text{ and } h_n : \mathbf{A} \otimes J_n \longrightarrow \mathbf{A} \otimes J_{n+1}, \text{ for } 0 \leq n \leq 2,$$

satisfying the following relations:

$$\partial_1 h_0 + h_{-1}\varepsilon = \mathrm{Id}_{\mathbf{A}} \text{ and } \partial_{n+1} h_n + h_{n-1}\partial_n = \mathrm{Id}_{\mathbf{A}\otimes J_n}, \text{ for } 0 \leq n \leq 2.$$

By assumption, the set X is finite. However, we will see that for the constructions of h_{-1}, h_0 and h_1 this hypothesis is not necessary.

From Proposition 2.2.6, every element f of $\mathrm{T}(V)$ admits a unique normal form for $\langle X \mid R\rangle$. This normal form is denoted by $\widehat{f}$.

For every $w \in \langle X\rangle$, we define $[w] \in A \otimes V$ as follows:

$$[w] = \begin{cases} 0, \text{ if } w \text{ is the empty word,} \\ \overline{w'} \otimes x, \text{ where } w' \in \langle X\rangle \text{ and } x \in X \text{ are such that } w = w'x. \end{cases}$$

The map $[\,] : \langle X\rangle \longrightarrow \mathbf{A} \otimes V$ is extended into a $\mathbb{K}$-linear map from $\mathrm{T}(V)$ to $\mathbf{A} \otimes V$. Let $w \in \langle X\rangle$ be a non empty word. For every $a \in \mathbf{A}$, the action of $\mathbf{A}$ on $[w]$ is given by $a.[w] = [fw]$, where $f \in \mathrm{T}(V)$ is such that $a = \overline{f}$.

In small homological degree, the Koszul complex of $\mathbf{A}$ is

$$\mathbf{A} \otimes \left(V \otimes \overline{R} \cap \overline{R} \otimes V\right) \xrightarrow{\partial_3} \mathbf{A} \otimes \overline{R} \xrightarrow{\partial_2} \mathbf{A} \otimes V \xrightarrow{\partial_1} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0,$$

where ∂_1 is defined by $\partial_1(1_{\mathbf{A}} \otimes v) = \overline{v}$ for every $v \in V$, ∂_2 is defined by $\partial_2(1_{\mathbf{A}} \otimes f) = [f]$ for every $f \in \overline{R}$ and ∂_3 is defined by $\partial_3(1_{\mathbf{A}} \otimes g) = \sum \overline{v} \otimes f$ where $\sum vf$ is a decomposition of $g \in V \otimes \overline{R} \cap \overline{R} \otimes V$ in $V \otimes \overline{R}$. By definition of ∂_3, $\partial_3(1_{\mathbf{A}} \otimes g)$ does not depend on the decomposition of g in $V \otimes \overline{R}$.

3.1.1. The constructions of h_{-1} and h_0. The maps $h_{-1} : \mathbb{K} \longrightarrow \mathbf{A}$ and $h_0 : \mathbf{A} \longrightarrow \mathbf{A} \otimes V$ are defined by

$$h_{-1}(1_{\mathbb{K}}) = 1_{\mathbf{A}} \text{ and } h_0(a) = \left[\widehat{f}\right], \text{ where } f \in \mathrm{T}(V) \text{ is such that } \overline{f} = a.$$

We have $h_0(1_{\mathbf{A}}) = 0$ and $h_{-1}\varepsilon(1_{\mathbf{A}}) = 1_{\mathbf{A}}$. If $\mathbf{A}$ belongs to $\mathbf{A}_m$ for $m \geq 1$, we have $\varepsilon(a) = 0$ and $\partial_1 h_0(a) = \overline{f}$. It follows that $\partial_1 h_0 + h_{-1}\varepsilon$ is equal to $\mathrm{Id}_{\mathbf{A}}$.

3.1.2. The construction of h_1. Recall from Proposition 2.2.6 that the algebra $\mathbf{A}$ admits as a basis the set $\{\overline{w},\ w \in \langle X\rangle \text{ is a normal form}\}$. Thus, in order to define $h_1 : \mathbf{A} \otimes V \longrightarrow \mathbf{A} \otimes R$, it is sufficient to define $h_1(\overline{w} \otimes x)$ for every normal form $w \in \langle X\rangle$ and every $x \in X$. Moreover, h_1 has to satisfy the relation

$$\partial_2 h_1(\overline{w} \otimes x) = \overline{w} \otimes x - h_0(\overline{wx}), \tag{1}$$

for every normal form $w \in \langle X\rangle$ and every $x \in X$.

We define $h_1(\overline{w} \otimes x)$ by Noetherian induction on wx. Assume that wx is a normal form. Then, let $h_1(\overline{w} \otimes x) = 0$. We have:

$$\begin{aligned} h_0(\overline{wx}) &= [\widehat{wx}] \\ &= [wx] \\ &= \overline{w} \otimes x. \end{aligned}$$

Thus, Relation 1 holds. Assume that wx is not a normal form and that $h_1\left(\overline{w'} \otimes x'\right)$ is defined and satisfies 1 for every normal form $w' \in \langle X\rangle$ and every $x' \in X$ such that $w'x' < wx$. The word wx can be written as a product w_1w_2, where $w_2 \in X^{(N)}$ is not a normal form. The presentation $\langle X \mid R\rangle$ is reduced. Thus, there exists a unique $f \in R$ such that $f = w_2 - \widehat{w_2}$. Let

$$h_1(\overline{w} \otimes x) = \overline{w_1} \otimes f + h_1([w_1\widehat{w_2}]).$$

We have:

$$\begin{aligned} \partial_2 h_1(\overline{w} \otimes x) &= [w_1 f] + \partial_2 h_1([w_1\widehat{w_2}]) \\ &= [w_1w_2] - [w_1\widehat{w_2}] + \partial_2 h_1([w_1\widehat{w_2}]). \end{aligned}$$

By induction hypothesis, $\partial_2 h_1([w_1\widehat{w_2}])$ is equal to $[w_1\widehat{w_2}] - \left[\widehat{w_1 w_2}\right]$. Hence, we have:

$$\begin{aligned}\partial_2 h_1(\overline{w}\otimes x) &= [w_1 w_2] - \left[\widehat{w_1 w_2}\right]\\ &= \overline{w}\otimes x - [\widehat{wx}]\\ &= \overline{w}\otimes x - h_0(\overline{wx}).\end{aligned}$$

Thus, Relation 1 holds.

3.1.3. Remark. We consider the $\mathbb{K}$-linear morphisms

$$\begin{aligned}&F_1 : \mathbf{A}\otimes V \longrightarrow V^{\otimes N},\ \overline{w_1 w_2}\otimes x \longmapsto \overline{w_1}\otimes w_2 x,\\ &F_1^1 : \mathbf{A}\otimes V^{\otimes N} \longrightarrow \mathbf{A}\otimes V,\ \overline{w_1}\otimes w_2 x \longmapsto \overline{w_1 w_2}\otimes x,\\ &F_2^1 : \mathbf{A}\otimes V \longrightarrow \mathbf{A}\otimes V^{\otimes N},\ \overline{w_1 w_2}\otimes x \longmapsto \overline{w_1}\otimes \widehat{w_2 x}.\end{aligned}$$

The inductive definition of h_1 implies that $h_1(\overline{w}\otimes x)$ is equal to

$$\left(F_1 - F_2^1\right)(\overline{w}\otimes x) + \left(F_1 - F_2^1\right)\left(F_1^1 F_2^1(\overline{w}\otimes x)\right) + \left(F_1 - F_2^1\right)\left(\left(F_1^1 F_2^1\right)^2(\overline{w}\otimes x)\right) + \cdots,$$

where $\left(F_1 - F_2^1\right)\left(\left(F_1^1 F_2^1\right)^{2k}(\overline{w}\otimes x)\right)$ vanishes for k sufficiently large.

In order to define h_2 we need the following:

3.1.4. Lemma. *Let $\mathbf{A}$ be an N-homogeneous algebra. Assume that $\mathbf{A}$ admits an extra-confluent presentation $\langle X \mid R\rangle$. Let $w_1 \in \langle X\rangle$, $w_2 \in X^{(N-1)}$ and $x_1,\ x_2 \in X$ such that:*

1. *$w_1 x_1$ and $x_1 w_2$ are normal forms for $\langle X \mid R\rangle$,*
2. *$w_2 x_2$ is not a normal form for $\langle X \mid R\rangle$.*

The word $w_1 x_1 w_2$ is a normal form for $\langle X \mid R\rangle$.

Proof. Assume that $w_1 x_1 w_2$ is not a normal form. By hypothesis, $w_1 x_1$ and $x_1 w_2$ are normal forms. Thus, there exist a right divisor u of w_1 and a left divisor v of w_2 such that $u x_1 v$ has length N and is not a normal form. In particular, $u x_1 w_2 x_2$ is the source of a critical branching. From Proposition 2.3.4, the word $x_1 w_2$ is not a normal form, which is a contradiction. Thus, Lemma 3.1.4 holds. □

3.1.5. The construction of h_2. Recall from Proposition 2.2.6 that the algebra $\mathbf{A}$ admits as a basis the set $\{\overline{w},\ w \in \langle X\rangle$ is a normal form$\}$. Thus, in order to define $h_2 : \mathbf{A}\otimes R \longrightarrow \mathbf{A}\otimes J_3$ it is sufficient to define $h_1(\overline{w}\otimes f)$ for every normal form $w \in \langle X\rangle$ and every $f \in R$. Moreover, h_2 has to satisfy the relation

$$\partial_3 h_2(\overline{w}\otimes f) = \overline{w}\otimes f - h_1\partial_2(\overline{w}\otimes f), \tag{2}$$

for every normal form $w \in \langle X\rangle$ and every $f \in R$.

We write $w = w_1 x_1$, $f = w' - \widehat{w'}$ and $w' = w_2 x_2$. We define $h_2(\overline{w}\otimes f)$ by Noetherian induction on $x_1 w_2$. Assume that $x_1 w_2$ is a normal form. Let $h_2(\overline{w}\otimes f) = 0$. We have:

$$\begin{aligned}h_1\partial_2(\overline{w}\otimes f) &= h_1([wf])\\ &= h_1([ww']) - h_1\left([w\widehat{w'}]\right)\\ &= h_1(\overline{ww_2}\otimes x') - h_1\left([w\widehat{w'}]\right).\end{aligned}$$

From Lemma 3.1.4, ww_2 is a normal form. Thus, by construction of h_1, we have:

$$h_1\left(\overline{ww_2}\otimes x'\right)=\overline{w}\otimes f+h_1\left(\widehat{[ww']}\right).$$

We conclude that $h_1\partial_2\left(\overline{w}\otimes f\right)$ is equal to $\overline{w}\otimes f$. Hence, Relation 2 holds.

Assume that $h_2\left(\overline{u}\otimes g\right)$ is defined and that (E_2) holds for every normal form $u\in\langle X\rangle$ and $g\in R$ such that $yv<x_1w_2$, where $y\in X$ and $v\in X^{(N-1)}$ are such that $u=u'y$ and $\mathrm{lm}\left(g\right)=vz$ for $u'\in\langle X\rangle$ and $z\in X$. We consider the two morphisms

$$S_1=S\otimes\mathrm{Id}_V \text{ and } S_2=\mathrm{Id}_V\otimes S.$$

The presentation $\langle X\mid R\rangle$ is side-confluent. Thus, from Lemma 2.2.7, there exists an integer k such that:

$$\langle\mathrm{Id}_{V^{\otimes N+1}}-S_2,\mathrm{Id}_{V^{\otimes N+1}}-S_1\rangle^k=\langle\mathrm{Id}_{V^{\otimes N+1}}-S_1,\mathrm{Id}_{V^{\otimes N+1}}-S_2\rangle^k.$$

We denote by Λ this common morphism. The image of Λ is included in $\ker\left(S_1\right)\cap\ker\left(S_2\right)$. The latter is equal to $\left(\overline{R}\otimes V\right)\cap\left(V\otimes\overline{R}\right)$. Recall that we have:

$$\langle\mathrm{Id}_{V^{\otimes N+1}}-S_2,\mathrm{Id}_{V^{\otimes N+1}}-S_1\rangle^k=\mathrm{Id}_{V^{\otimes N+1}}+\sum_{i=1}^{k-1}(-1)^i\left(\langle S_1,S_2\rangle^i+\langle S_2,S_1\rangle^i\right)+(-1)^k\langle S_2,S_1\rangle^k.$$

Thus, we have:

$$\Lambda=\left(\mathrm{Id}_{V^{\otimes N+1}}-S_2\right)+\left(\mathrm{Id}_{V^{\otimes N+1}}-S_2\right)\sum_{i=1}^{k-1}(-1)^i\ g_i\left(S_1,S_2\right),$$

where $g_i\left(S_1,S_2\right)$ denotes the product $S_1S_2S_1\cdots$ with i factors. In particular, there exist $f_1,\cdots,f_l\in R$, $x_1,\cdots,x_l\in X$ and $\lambda_1,\cdots,\lambda_l\in\mathbb{K}$ such that $x_iw_i<x_1w_2$ where $\mathrm{lm}\left(f_i\right)=w_iy_i$ and

$$\Lambda\left(xw'\right)=xf+\sum_{i=1}^{l}\lambda_ix_if_i.$$

Then, let

$$h_2\left(\overline{w}\otimes f\right)=\overline{w_1}\otimes\Lambda\left(xw'\right)-\sum_{i=1}^{l}\lambda_ih_2\left(\overline{w_1x_i}\otimes f_i\right).$$

We will show in Section 4 that Relation 2 holds.

3.1.6. Remark. We consider the $\mathbb{K}$-linear maps

$$\begin{aligned}
&F_2:\mathbf{A}\otimes V^{\otimes N}\longrightarrow V^{\otimes N+1},\ \overline{w_1x}\otimes w_2\longmapsto\overline{w_1}\otimes xw_2,\\
&F_1^2:\mathbf{A}\otimes V^{\otimes N+1}\longrightarrow\mathbf{A}\otimes V^{\otimes N},\ \overline{w_1}\otimes xw_2\longmapsto\overline{w_1x}\otimes w_2,\\
&F_2^2:\mathbf{A}\otimes V^{\otimes N}\longrightarrow\mathbf{A}\otimes V^{\otimes N+1},\ \overline{w_1}x\otimes w_2\longmapsto\overline{w_1}\otimes xw_2-\Lambda\left(xw_2\right).
\end{aligned}$$

The inductive definition of h_2 implies that $h_2\left(\overline{w}\otimes f\right)$ is equal to

$$\left(F_2-F_2^2\right)\left(\overline{w}\otimes f\right)+\left(F_2-F_2^2\right)\left(F_2^1F_2^2\left(\overline{w}\otimes f\right)\right)+\left(F_2-F_2^2\right)\left(\left(F_2^1F_2^2\right)^2\left(\overline{w}\otimes f\right)\right)+\cdots,$$

where $\left(F_2-F_2^2\right)\left(\left(F_2^1F_2^2\right)^{2k}\left(\overline{w}\otimes f\right)\right)$ vanishes for k sufficiently large.

3.1.7. Example. The construction of our contracting homotopy for the Koszul complex of the enveloping algebra of the Heisenberg Lie algebra is done in Section 5.3.

3.2 Reduction operators and confluence algebras

We fix a finite set Y, totally ordered by a relation $<$. For every $v \in \mathbb{K}Y \setminus \{0\}$, we denote by $\mathrm{lm}(v)$ the greatest element of Y occurring in the decomposition of v. We extend the order $<$ to a partial order on $\mathbb{K}Y$ in the following way: we have $v < w$ if either $v = 0$ or if $\mathrm{lm}(v) < \mathrm{lm}(w)$.

In this section we recall some results from [Ber98] about reduction operators and confluence algebras.

3.2.1. Reduction operators. A linear projector T of $\mathbb{K}Y$ is called a *reduction operator relatively to* Y if for every $y \in Y$, we have either $T(y) = y$ or $T(y) < y$. We denote by $\mathrm{Red}\,(Y)$ the set of reduction operators relatively to Y.

3.2.2. Lattice structure. The set $\mathrm{Red}\,(Y)$ admits a lattice structure. To define the order, recall from [Ber98, Lemma 2.2] that if $U,\ T \in \mathrm{Red}\,(Y)$ are such that $\ker(U)$ is included in $\ker(T)$, then $\mathrm{im}\,(T)$ is included in $\mathrm{im}\,(U)$. Thus, the relation defined by $T \preceq U$ if $\ker(U) \subset \ker(T)$ is an order relation on $\mathrm{Red}\,(Y)$.

We denote by $\mathscr{L}(\mathbb{K}Y)$ the lattice of sub-vector spaces of $\mathbb{K}Y$: the order is the inclusion, the lower bound is the intersection and the upper bound is the sum. To define the upper bound and the lower bound on $\mathrm{Red}\,(Y)$, recall from [Ber98, Theorem 2.3] that the map

$$\begin{aligned} \theta_Y \colon \mathrm{Red}\,(Y) &\longrightarrow \mathscr{L}(\mathbb{K}Y), \\ T &\longmapsto \ker(T) \end{aligned}$$

is a bijection. The lower bound $T_1 \wedge T_2$ and the upper bound $T_1 \vee T_2$ of two elements T_1 and T_2 of $\mathrm{Red}\,(Y)$ are defined in the following way:

$$\begin{cases} T_1 \wedge T_2 = \theta_Y^{-1}\left(\ker(T_1) + \ker(T_2)\right), \\ T_1 \vee T_2 = \theta_Y^{-1}\left(\ker(T_1) \cap \ker(T_2)\right). \end{cases}$$

3.2.3. Remark. The lattice $\mathrm{Red}\,(Y)$ admits $\mathrm{Id}_{\mathbb{K}Y}$ as maximum and $0_{\mathbb{K}Y}$ as minimum.

3.2.4. Confluent pairs of reduction operators. A pair $P = (T_1, T_2)$ of reduction operators relatively to Y is said to be *confluent* if there exists an integer k such that:

$$\langle T_1, T_2 \rangle^k = \langle T_2, T_1 \rangle^k .$$

We will see in Section 3.3 the link between this notion and the side-confluent presentations.

3.2.5. Confluence algebras. Let k be an integer. The *confluence algebra of degree* k is the algebra presented by

$$\left\langle s_1, s_2 \mid \ s_i^2 = s_i,\ \langle s_1, s_2 \rangle^k = \langle s_2, s_1 \rangle^k ,\ i = 1, 2 \right\rangle .$$

This algebra is denoted by $\mathscr{A}_k$. Let us consider the following elements of $\mathscr{A}_k$:

$$\begin{aligned} \sigma &= \langle s_1, s_2 \rangle^k = \langle s_2, s_1 \rangle^k , \\ \gamma_1 &= (1 - s_2) \sum_{i \in I} \langle s_2, s_1 \rangle^i , \\ \gamma_2 &= (1 - s_1) \sum_{i \in I} \langle s_1, s_2 \rangle^i , \\ \lambda &= 1 - (\sigma + \gamma_1 + \gamma_2) , \end{aligned}$$

where I is the set of odd integers between 1 and $k-1$. We easily check that we have the following relations:

$$\gamma_i s_i = \gamma_i, \text{ for } i = 1,\ 2, \tag{3a}$$

$$s_i \gamma_i = s_i - \sigma, \text{ for } i = 1,\ 2. \tag{3b}$$

3.2.6. P-representations of confluence algebras. Let $P = (T_1, T_2)$ be a confluent pair of reduction operators relatively to Y. Let k be an integer such that $\langle T_1, T_2 \rangle^k = \langle T_2, T_1 \rangle^k$. We consider the morphism of algebras

$$\begin{aligned} \varphi^P \colon \mathscr{A}_k &\longrightarrow \operatorname{End}(\mathbb{K}Y)\,. \\ s_i &\longmapsto T_i \end{aligned}$$

The morphism φ^P is called the *P-representation of $\mathscr{A}_k$*. Recall from [Ber98] that:

$$\varphi^P(\sigma) = T_1 \wedge T_2, \tag{4a}$$

$$\varphi^P(1-\lambda) = T_1 \vee T_2. \tag{4b}$$

3.2.7. The left bound and the right bound. Let $P = (T_1, T_2)$ be a confluent pair of reduction operators relatively to Y. By definition of λ and from 3.2.6, we have:

$$T_1 \vee T_2 = T_1 \wedge T_2 + \varphi^P(\gamma_1) + \varphi^P(\gamma_2)\,. \tag{5}$$

The morphisms $\varphi^P(\gamma_1)$ and $\varphi^P(\gamma_2)$ are called the *left bound* of P and the *right bound of* P, respectively.

We end this section with the following:

3.2.8. Lemma. *Let $P = (T_1, T_2)$ be a confluent pair of reduction operators relatively to Y. Let W be a sub-vector space of $\mathbb{K}Y$. If W is included in* $\ker(T_i)$ *for $i = 1$ or 2, we have:*

$$\varphi^P(\gamma_i)_{|W} = T_1 \vee T_{2|W}.$$

Proof. By definition, σ and γ_i factorize on the right by s_i. Hence, the restrictions of $\varphi^P(\sigma)$ and $\varphi^P(\gamma_i)$ to W vanish. Thus, Lemma 3.2.8 is a consequence of Relation 5. □

3.3 Reduction operators and side-confluent presentations

Let $\mathbf{A}$ be an N-homogeneous algebra. We suppose that $\mathbf{A}$ admits a side-confluent presentation $\langle X \mid R \rangle$ where X is a totally ordered finite set. For every integer m, the set $X^{(m)}$ is finite and totally ordered for the lexicographic order induced by the order over X. We write $V = \mathbb{K}X$.

3.3.1. Normal forms and the Koszul complex. In Lemma 3.3.3 we will link together the Koszul complex of $\mathbf{A}$ and the reduction operators. In this way, recall from Proposition 2.2.6 that every element $f \in \mathrm{T}(V)$ admits a unique normal form for $\langle X \mid R \rangle$, denoted by $\widehat{f}$. Let

$$\begin{aligned} \phi \colon \mathrm{T}(V) &\longrightarrow \mathrm{T}(V)\,. \\ f &\longmapsto \widehat{f} \end{aligned}$$

Recall from Proposition 2.2.6 that for every $f \in \mathrm{T}(V)$, we have $f \in I(R)$ if and only if $\widehat{f} = 0$. Hence, ϕ induces a $\mathbb{K}$-linear isomorphism $\overline{\phi}$ from $\mathbf{A}$ to $\operatorname{im}(\phi)$. In particular, for every integer n, the morphism $\phi_n = \overline{\phi} \otimes \mathrm{Id}_{V \otimes l_N(n)}$ is a $\mathbb{K}$-linear isomorphism from $\mathbf{A} \otimes J_n$ to $\operatorname{im}(\phi) \otimes J_n$. Thus, the Koszul complex $(K_\bullet, \partial)$ of $\mathbf{A}$ is isomorphic to the complex of vector spaces $(K'_\bullet, \partial')$

$$\cdots \xrightarrow{\partial'_{n+1}} \operatorname{im}(\phi) \otimes J_n \xrightarrow{\partial'_n} \operatorname{im}(\phi) \otimes J_{n-1} \longrightarrow \cdots \xrightarrow{\partial'_2} \operatorname{im}(\phi) \otimes J_1 \xrightarrow{\partial'_1} \operatorname{im}(\phi) \xrightarrow{\varepsilon'} \mathbb{K} \longrightarrow 0,$$

where ∂'_n is equal to $\phi_{n-1} \circ \partial_n \circ \phi_n^{-1}$.

3.3.2. Definition. The complex $(K'_\bullet, \partial')$ is the *normalised Koszul complex of* $\mathbf{A}$.

3.3.3. Lemma.

1. *For every integer m, the restriction of ϕ to $V^{\otimes m}$ is a reduction operator relatively to $X^{(m)}$ and its kernel is equal to $I(R)_m$.*
2. *Let n be an integer such that $n \geq 1$. The morphism ∂'_n is the restriction to* $\operatorname{im}(\phi) \otimes J_n$ *of the morphism $\varphi_n : \bigoplus_{m \geq l_N(n)} V^{\otimes m} \longrightarrow \mathrm{T}(V)$ defined by*

$$\varphi_{n|V^{\otimes m}} = \phi_{|V^{\otimes m - l_N(n-1)}} \otimes \mathrm{Id}_{V^{\otimes l_N(n-1)}}.$$

Proof. Let us show the first point. The presentation $\langle X \mid R \rangle$ is N-homogeneous. Thus, for every $w \in X^{(m)}$, $\phi(w)$ belongs to $V^{\otimes m}$. In particular, the restriction of ϕ to $V^{\otimes m}$ is an endomorphism of $V^{\otimes m}$. Let $w \in X^{(m)}$. If w is a normal form, then $\phi(w)$ is equal to w. In particular, $\phi_{|V^{\otimes m}}$ is a projector. If w is not a normal form, then $\phi(w) = \widehat{w}$ is strictly smaller than w. Thus, $\phi_{|V^{\otimes m}}$ is a reduction operator relatively to $X^{(m)}$. Moreover, $\widehat{f}$ is equal to 0 if and only if f belongs to $I(R)$. Thus, the kernel of $\phi_{|V^{\otimes m}}$ is equal to $I(R)_m$.

Let us show the second point. Recall from 2.1.2 that the differential $\partial_n : \mathbf{A} \otimes J_n \longrightarrow \mathbf{A} \otimes J_{n-1}$ of the Koszul complex of $\mathbf{A}$ is the restriction to $\mathbf{A} \otimes J_n$ of the $\mathbf{A}$-linear map defined by:

$$\mathbf{A} \otimes V^{\otimes l_N(n)} \longrightarrow \mathbf{A} \otimes V^{\otimes l_N(n-1)},$$

$$1_{\mathbf{A}} \otimes w \longmapsto \overline{w_1} \otimes w_2$$

where $w_1 \in X^{(l_N(n) - l_N(n-1))}$ and $w_2 \in X^{(l_N(n-1))}$ are such that $w = w_1 w_2$. Thus, the map ∂'_n is the restriction of the morphism which maps a word w of length $m \geq l_N(n)$ to $\widehat{w_1} w_2$, where $w_1 \in X^{(m - l_N(n-1))}$ and $w_2 \in X^{(l_N(n-1))}$ are such that $w = w_1 w_2$. The latter is equal to $\phi_{|V^{\otimes m - l_N(n-1)}} \otimes \mathrm{Id}_{V^{\otimes l_N(n-1)}}$. □

3.3.4. Lattice properties. Let $S \in \operatorname{End}\left(V^{\otimes N}\right)$ be the operator of $\langle X \mid R \rangle$:

$$S(w) = \begin{cases} \operatorname{lm}(f) - f, & \text{if there exists } f \in R \text{ such that } w = \operatorname{lm}(f), \\ w, & \text{otherwise.} \end{cases}$$

The properties of S described in 2.2.4 imply that S is equal to $\theta^{-1}_{X^{(N)}}\left(\overline{R}\right)$. For every integers m and i such that $m \geq N$ and $0 \leq i \leq m - N$, we consider the following reduction operator relatively to $X^{(m)}$:

$$S_i^{(m)} = \mathrm{Id}_{V^{\otimes i}} \otimes S \otimes \mathrm{Id}_{V^{\otimes m-N-i}}.$$

The kernel of $S_i^{(m)}$ is equal to $V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-N-i}$.

The presentation $\langle X \mid R \rangle$ is side-confluent. Hence, the pair $\left(S_i^{(2N-1)}, S_j^{(2N-1)}\right)$ is confluent for every integers i and j such that $0 \leq i, j \leq N - 1$. We deduce from [Ber01, Section 3] and [Ber98, Theorem 2.12] that for every integer m such that $m \geq N$, the sub-lattice of $\operatorname{Red}\left(X^{(m)}\right)$ spanned by $S_0^{(m)}, \cdots, S_{m-N}^{(m)}$ is *confluent* (that is, the elements of this lattice are pairwise confluent) and *distributive* (that is, for every S, T, U belonging to this lattice, we have $(S \wedge T) \vee U = (S \vee U) \wedge (T \vee U)$).

4 The left bound of a side-confluent presentation

Through this section we fix an N-homogeneous algebra $\mathbf{A}$. We assume that $\mathbf{A}$ admits an N-homogeneous presentation $\langle X \mid R \rangle$ where X is a totally ordered finite set. This presentation is also fixed. We write $V = \mathbb{K}X$. We consider the notations of 3.3.4.

4.1 Reduction pairs associated with a presentation

For every integers n and m such that $m \geq l_N(n)$, we consider the following reduction operators relatively to $X^{(m)}$:

$$F_1^{n,m} = \theta_{X^{(m)}}^{-1}\left(I(R)_{m-l_N(n)} \otimes V^{\otimes l_N(n)}\right),$$

$$F_2^{n,m} = \begin{cases} \mathrm{Id}_{V^{\otimes m}}, \text{ if } m < l_N(n+1), \\ \theta_{X^{(m)}}^{-1}\left(V^{\otimes m-l_N(n+1)} \otimes J_{n+1}\right), \text{ otherwise.} \end{cases}$$

The pair $(F_1^{n,m}, F_2^{n,m})$ is denoted by $P_{n,m}$.

4.1.1. Definition. The pair $P_{n,m}$ is the *reduction pair of bi-degree (n,m) associated with* $\langle X \mid R\rangle$.

4.1.2. Lemma. *Let n and m be two integers such that $n \geq 1$ and $l_N(n) \leq m < l_N(n+1)$. Then, $m - l_N(n-1)$ is smaller than $N-1$ and $F_1^{n-1,m}$ is equal to* $\mathrm{Id}_{V^{\otimes m}}$.

Proof. First, we show that $m - l_N(n-1)$ is smaller than $N-1$. Assume that m is a multiple of N: $m = kN$. In this case, the hypothesis $l_N(n) \leq m < l_N(n+1)$ implies that n is equal to $2k$. Thus, $l_N(n-1)$ is equal to $(k-1)N+1$. That implies that $m - l_N(n-1)$ is equal to $N-1$. Assume that m is not a multiple of N: $m = kN+r$ with $1 \leq r \leq N-1$. In this case, the hypothesis $l_N(n) \leq m < l_N(n+1)$ implies that n is equal to $2k+1$. Thus, $m - l_N(n-1) = m - kN$ is smaller than $N-1$.

Let us show that $F_1^{n-1,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. The first part of the lemma implies that $I(R)_{m-l_N(n-1)}$ is equal to $\{0\}$. Thus, the kernel of $F_1^{n-1,m}$ is equal to $\{0\}$, that is, $F_1^{n-1,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. □

4.1.3. Lemma.

1. *Let n and let m be two integers such that $m \geq l_N(n+2)$. We have:*

$$F_1^{n,m} = S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+2)}^{(m)}.$$

2. *Let n and m be two integers such that $n \geq 1$ and $m \geq l_N(n+1)$. We have:*

$$F_2^{n,m} = S_{m-l_N(n+1)}^{(m)} \vee \cdots \vee S_{m-N}^{(m)}.$$

Proof. By definition of $\wedge$, we have:

$$\begin{aligned}
\ker\left(S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+2)}^{(m)}\right) &= \sum_{i=0}^{m-l_N(n+2)} \ker\left(S_i^{(m)}\right) \\
&= \sum_{i=0}^{m-l_N(n+2)} V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-N-i} \\
&= \left(\sum_{i=0}^{m-l_N(n+2)} V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-l_N(n)-N-i}\right) \otimes V^{\otimes l_N(n)} \\
&= \left(\sum_{i=0}^{m-l_N(n)-N} V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-l_N(n)-N-i}\right) \otimes V^{\otimes l_N(n)} \\
&= I(E)_{m-l_N(n)} \otimes V^{\otimes l_N(n)}.
\end{aligned}$$

By definition of $\vee$, we have:

$$\begin{aligned}
\ker\left(S^{(m)}_{m-l_N(n+1)}\vee\cdots\vee S^{(m)}_{m-N}\right) &= \bigcap_{i=m-l_N(n+1)}^{m-N}\ker\left(S^{(m)}_i\right)\\
&= \bigcap_{i=m-l_N(n+1)}^{m-N} V^{\otimes i}\otimes\overline{R}\otimes V^{\otimes m-N-i}\\
&= V^{\otimes m-l_N(n+1)}\otimes\left(\bigcap_{i=0}^{l_N(n+1)-N} V^{\otimes i}\otimes\overline{R}\otimes V^{\otimes l_N(n+1)-N-i}\right)\\
&= V^{\otimes m-l_N(n+1)}\otimes J_{n+1}.
\end{aligned}$$

The map $\theta_{X^{(m)}}$ being a bijection, the two relations hold. □

4.1.4. Theorem. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra admitting a side-confluent presentation* $\langle X \mid R\rangle$*, where* X *is a finite set. The reduction pairs associated with* $\langle X \mid R\rangle$ *are confluent.*

Proof. Let n and m be two integers such that $m \geq l_N(n)$. We have to show that the reduction pair of bi-degree (n, m) associated with $\langle X \mid R\rangle$ is confluent. We proceed in four steps.

Step 1. Assume that $n = 0$. We have $P_{0,0} = (\mathrm{Id}_{\mathbb{K}}, \mathrm{Id}_{\mathbb{K}})$. Thus, the pair $P_{0,0}$ is confluent. Let m be an integer such that $m \geq 1$. The kernel of $F^{0,m}_2$ is equal to $V^{\otimes m-1}\otimes J_1 = V^{\otimes m}$. Thus, $F^{0,m}_2$ is equal to $0_{V^{\otimes m}}$. In particular, the operators $F^{0,m}_1$ and $F^{0,m}_2$ commute, that is, they satisfy the relation $\left\langle F^{0,m}_1, F^{0,m}_2\right\rangle^2 = \left\langle F^{0,m}_2, F^{0,m}_1\right\rangle^2$. Hence, the pair $P_{0,m}$ is confluent for every integer m.

Step 2. Assume that $n \geq 1$ and $l_n(n) \leq m < l_N(n+1)$. The pair $P_{n,m}$ is equal to $(F^{n,m}_1, \mathrm{Id}_{V^{\otimes m}})$. Thus, the operators $F^{n,m}_1$ and $F^{n,m}_2$ commute. We conclude that the pairs $P_{n,m}$ such that $n \geq 1$ and $l_n(n) \leq m < l_N(n+1)$ are confluent.

Step 3. Assume that $n \geq 1$ and $l_N(n+1) \leq m < l_N(n+2)$. From Lemma 4.1.2, the morphism $F^{n,m}_1$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. In particular, the operators $F^{n,m}_1$ and $F^{n,m}_2$ commute. Thus, the pairs $P_{n,m}$ such that $n \geq 1$ and $l_N(n+1) \leq m < l_N(n+2)$ are confluent.

Step 4. Assume that $n \geq 1$ and $m \geq l_N(n+2)$. Lemma 4.1.3 implies that $F^{n,m}_1$ and $F^{n,m}_2$ belong to the lattice generated by $S^{(m)}_i$, for $0 \leq i \leq m-N$. From 3.3.4 the latter is confluent. Hence, the pairs $P_{n,m}$ such that $n \geq 1$ and $m \geq l_N(n+2)$ are confluent. □

4.2 Construction

Through this section, we assume that the presentation $\langle X \mid R\rangle$ of $\mathbf{A}$ is side-confluent. From Proposition 2.2.6, every element f of $\mathrm{T}(V)$ admits a unique normal for $\langle X \mid R\rangle$. This normal form is denoted by $\widehat{f}$. We denote by ϕ the endomorphism of $\mathrm{T}(V)$ which maps an element to its unique normal form. We consider the notations of Section 4.1.

4.2.1. Lemma. *For every integers* n *and* m *such that* $m \geq l_N(n)$*, the operator* $F^{n,m}_1$ *is equal to* $\phi_{|V^{\otimes m-l_N(n)}}\otimes\mathrm{Id}_{V^{\otimes l_N(n)}}$.

Proof. From Point 1 of Lemma 3.3.3, the operator $\phi_{|V^{\otimes m-l_N(n)}}\otimes\mathrm{Id}_{V^{\otimes l_N(n)}}$ is a reduction operator relatively to $X^{(m)}$ and its kernel is equal to $I(R)_{m-l_n(n)}\otimes V^{\otimes l_n(n)}$. The map $\theta_{X^{(m)}}$ being a bijection, Lemma 4.2.1 holds. □

4.2.2. Lemma. *Let n be an integer. Let $h'_n : \bigoplus_{m \geq l_N(n)} V^{\otimes m} \longrightarrow \mathrm{T}(V)$ be the $\mathbb{K}$-linear map defined by*

$$h'_{n|V^{\otimes m}} = \varphi^{P_{n,m}}(\gamma_1),$$

where $\varphi^{P_{n,m}}(\gamma_1)$ is the left bound of $P_{n,m}$. The image of h'_n is included in $\mathrm{im}(\phi) \otimes J_{n+1}$.

Proof. Let m be an integer such that $m \geq l_N(n)$. By definition of the left bound, there exists an endomorphism H of $V^{\otimes m}$ such that

$$\varphi^{P_{n,m}}(\gamma_1) = \left(\mathrm{Id}_{V^{\otimes m}} - F_2^{n,m}\right) F_1^{n,m} H.$$

The image of $F_1^{n,m} = \phi_{|V^{\otimes m - l_N(n)}} \otimes V^{\otimes l_N(n)}$ is equal to the vector space spanned by the elements with shape $w_1 w_2$ where $w_1 \in X^{(m - l_N(n))}$ is a normal form and $w_2 \in X^{(l_N(n))}$.

Let

$$G = \theta^{-1}_{X^{(l_N(n+1))}}(J_{n+1}).$$

We have $F_2^{n,m} = \mathrm{Id}_{V^{\otimes m - l_N(n+1)}} \otimes G$. The latter implies that

$$\left(\mathrm{Id}_{V^{\otimes m}} - F_2^{n,m}\right) = \mathrm{Id}_{V^{\otimes m - l_N(n+1)}} \otimes \left(\mathrm{Id}_{V^{\otimes l_N(n+1)}} - G\right).$$

We conclude that the image of $\varphi^{P_{n,m}}(\gamma_1)$ is included in the vector space spanned by elements with shape wf where $w \in X^{(m - l_N(n+1))}$ is a normal form and $f \in J_{n+1}$. This vector space is equal to $\mathrm{im}\left(\phi_{|V^{\otimes m - l_N(n+1)}}\right) \otimes J_{n+1}$. □

4.2.3. Definition. For every integer n, let

$$h_n = \phi_{n+1}^{-1} \circ h'_n \circ \phi_n : \mathbf{A} \otimes J_n \longrightarrow \mathbf{A} \otimes J_{n+1},$$

where ϕ_n is the $\mathbb{K}$-linear isomorphism between $\mathbf{A} \otimes J_n$ and $\mathrm{im}(\phi) \otimes J_n$ defined in 3.3.1. The family $(h_n)_n$ is the *left bound of* $\langle X \mid R \rangle$.

4.2.4. Reduction relations. Let n and m be two integers such that $m \geq l_N(n)$. Then, we denote by $K_n^{(m)} = \mathrm{im}\left(\phi_{|V^{\otimes m - l_N(n)}}\right) \otimes J_n$. In particular, we have:

$$\mathrm{im}(\phi) \otimes J_n = \bigoplus_{m \geq l_N(n)} K_n^{(m)}.$$

We say that the presentation $\langle X \mid R \rangle$ satisfy the *reduction relations* if for every integers n and m such that $m \geq l_n(n)$, the following equality holds:

$$(r_{n,m}) \quad F_1^{n,m} \wedge F_2^{n,m}{}_{|K_n^{(m)}} = F_1^{n-1,m} \vee F_2^{n-1,m}{}_{|K_n^{(m)}}.$$

4.2.5. Proposition. *Let $\mathbf{A}$ be an N-homogeneous algebra. Assume that $\mathbf{A}$ admits a side-confluent presentation $\langle X \mid R \rangle$ where X is a finite set. The left bound of $\langle X \mid R \rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$ if and only if $\langle X \mid R \rangle$ satisfies the reduction relations.*

Proof. The left bound of $\langle X \mid R \rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$ if and only if the family $\left(h'_n : \mathrm{im}(\phi) \otimes J_n \longrightarrow \mathrm{im}(\phi) \otimes J_{n+1}\right)_n$ defined in Lemma 4.2.2 is a contracting homotopy for the normalised Koszul complex of $\mathbf{A}$.

From Point 2 of Lemma 3.3.3, the restriction of $F_1^{n-1,m} = \phi_{|V^{\otimes m-l_N(n-1)}} \otimes \mathrm{Id}_{V^{\otimes l_n(n-1)}}$ to $K_n^{(m)}$ is equal to the restriction of ∂'_n to $K_n^{(m)}$. Thus, the family $(h'_n)_n$ is a contracting homotopy for $(K'_\bullet, \partial')$ if and only if for every n and m such that $n \geq 1$ and $m \geq l_N(n)$, the following relation holds:

$$\left(\varphi^{P_{n,m}}(s_1)\,\varphi^{P_{n,m}}(\gamma_1) + \varphi^{P_{n-1,m}}(\gamma_1)\,\varphi^{P_{n-1,m}}(s_1)\right)_{|K_n^{(m)}} = \mathrm{Id}_{K_n^{(m)}}.$$

From Relation 3b (see page 15) and Relation 4a (see page 15), we have:

$$\begin{aligned}\varphi^{P_{n,m}}(s_1)\,\varphi^{P_{n,m}}(\gamma_1) &= F_1^{n,m} - \varphi^{P_{n,m}}(\sigma)\\ &= F_1^{n,m} - F_1^{n,m} \wedge F_2^{n,m}.\end{aligned}$$

The image of $F_1^{n,m} = \phi_{|V^{\otimes m-l_N(n)}} \otimes \mathrm{Id}_{V^{\otimes l_N(n)}}$ is equal to $\mathrm{im}\left(\phi_{|V^{\otimes m-l_N(n)}}\right) \otimes V^{\otimes l_N(n)}$. Thus, $K_n^{(m)}$ is included in $\mathrm{im}\,(F_1^{n,m})$. In particular, the restriction of $F_1^{n,m}$ to $K_n^{(m)}$ is the identity map. We deduce that the left bound family of $\langle X \mid R\rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$ if and only if the following relation holds:

$$\left(\varphi^{P_{n-1,m}}(\gamma_1)\,\varphi^{P_{n-1,m}}(s_1)\right)_{|K_n^{(m)}} = F_1^{n,m} \wedge F_2^{n,m}{}_{|K_n^{(m)}}.$$

From Relation 3a (see page 15), $\varphi^{P_{n-1,m}}(\gamma_1)\,\varphi^{P_{n-1,m}}(s_1)$ is equal to $\varphi^{P_{n-1,m}}(\gamma_1)$. Thus, it is sufficient to show:

$$\varphi^{P_{n-1,m}}(\gamma_1)_{|K_n^{(m)}} = F_1^{n-1,m} \vee F_2^{n-1,m}{}_{|K_n^{(m)}}. \tag{6}$$

By construction, $K_n^{(m)}$ is included in $\ker\left(F_2^{n-1,m}\right)$. Hence, Relation 6 is a consequence of Lemma 3.2.8. $\square$

The following lemma will be used in the proof of Theorem 4.3.5:

4.2.6. Lemma. *Let n and m be two integers such that $n \geq 1$ and $l_N(n) \leq m < l_N(n+1)$. The operators $F_1^{n,m}$ and $F_1^{n-1,m} \vee F_2^{n-1,m}$ commute.*

Proof. The pair $P_{n,m}$ being confluent, we deduce from Relation 4b (see page 15) that $F_1^{n-1,m} \vee F_2^{n-1,m}$ is polynomial in $F_1^{n-1,m}$ and $F_2^{n-1,m}$. Hence, it is sufficient to show that $F_1^{n,m}$ commutes with $F_1^{n-1,m}$ and $F_2^{n-1,m}$.

Let

$$G = \theta^{-1}_{X^{(l_N(n))}}(J_n).$$

We have $F_2^{n-1,m} = \mathrm{Id}_{V^{\otimes m-l_N(n)}} \otimes G$. Thus, $F_1^{n,m} = \phi_{|V^{\otimes m-l_N(n)}} \otimes \mathrm{Id}_{V^{\otimes l_N(n)}}$ commutes with $F_2^{n,m}$. Moreover, the morphism $F_1^{n,m}$ (respectively $F_1^{n-1,m}$) maps a word w of length m to $\widehat{w_1}w_2$ (respectively $\widehat{w'_1}w'_2$), where $w_1 \in X^{(m-l_N(n))}$ and $w_2 \in X^{(l_N(n))}$ (respectively $w'_1 \in X^{(m-l_N(n-1))}$ and $w'_2 \in X^{(l_N(n-1))}$) are such that $w = w_1w_2$ (respectively $w = w'_1w'_2$). Thus, the two compositions $F_1^{n,m}F_1^{n-1,m}$ and $F_1^{n-1,m}F_1^{n,m}$ are equal to $F_1^{n-1,m}$. $\square$

4.3 Extra-confluent presentations and reduction relations

Through this section we assume that the presentation $\langle X \mid R\rangle$ is extra-confluent. Our aim is to show that $\langle X \mid R\rangle$ satisfies the reduction relations. In this way, we will show in Proposition 4.3.4 that the extra-condition enables us to link together the reduction pairs associated with $\langle X \mid R\rangle$. We consider the notations of Section 4.1.

4.3.1. Lemma. *Let m, r and k be three integers such that $m \geq N+2$, $2 \leq k \leq N-1$ and $r+k \leq m-N$. Then, we have:*

1. $S_r^{(m)} \vee S_{r+k}^{(m)} = S_r^{(m)} \vee \cdots \vee S_{r+k}^{(m)}$,
2. $\left(S_r^{(m)} \wedge \cdots \wedge S_{r+k-1}^{(m)}\right) \vee S_{r+k}^{(m)} = S_{r+k-1}^{(m)} \vee S_{r+k}^{(m)}$.

Proof. Let us prove the point 1. The extra-condition implies the following inclusion:

$$\left(V^{\otimes r+k} \otimes \overline{R} \otimes V^{\otimes m-N-r-k}\right) \cap \left(V^{\otimes r} \otimes \overline{R} \otimes V^{\otimes m-N-r}\right) \ \subset \ V^{\otimes r+k-1} \otimes \overline{R} \otimes V^{\otimes m-N-r-k+1}.$$

Applying the bijection $\theta_{X^{(m)}}^{-1}$, we have:

$$S_{r+k-1}^{(m)} \preceq S_r^{(m)} \vee S_{r+k}^{(m)}.$$

By definition of the upper bound, we deduce that $S_r^{(m)} \vee S_{r+k-1}^{(m)} \vee S_{r+k}^{(m)}$ is equal to $S_r^{(m)} \vee S_{r+k}^{(m)}$. By induction on k, we obtain the first relation.

Let us prove the point 2. Recall from 3.3.4 that the lattice spanned by $S_0^{(m)}, \cdots, S_{m-N}^{(m)}$ is distributive. Thus, the left hand side of 2 is equal to $\left(S_r^{(m)} \vee S_{r+k}^{(m)}\right) \wedge \cdots \wedge \left(S_{r+k-1}^{(m)} \vee S_{r+k}^{(m)}\right)$. By the first point, for every integer i such that $0 \leq i \leq n-2$, $S_{r+i}^{(m)} \vee S_{r+k}^{(m)}$ is equal to $S_{r+i}^{(m)} \vee \cdots \vee S_{r+k}^{(m)}$, so it is greater than $S_{r+k-1}^{(m)} \vee S_{k+r}^{(m)}$. By definition of the lower bound, the second relation holds. □

4.3.2. Lemma. *Let n and m be two integers such that $n \geq 2$ and $l_N(n+1) \leq m < l_N(n+2)$. We have:*

$$\left(S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)}\right) \vee S_{m-l_N(n)}^{(m)} = S_{m-l_N(n+1)}^{(m)} \vee \cdots \vee S_{m-l_N(n)}^{(m)}. \tag{7}$$

Proof. From Lemma 4.1.2, the hypothesis $l_N(n+1) \leq m < l_N(n+2)$ implies that $m - l_N(n)$ is smaller than $N-1$.

Assume that m is a multiple of N. The hypothesis $l_N(n+1) \leq m < l_N(n+2)$ implies that m is equal to $l_N(n+1)$. Thus, the left hand side of 7 is equal to $S_0^{(m)} \vee S_{m-l_N(n)}^{(m)}$ and the right hand side of 7 is equal to $S_0^{(m)} \vee \cdots \vee S_{m-l_N(n)}^{(m)}$. Hence, Relation 7 is a consequence of Lemma 4.3.1 point 1.

Assume that m is not a multiple of N. The hypothesis $l_N(n+1) \leq m < l_N(n+2)$ implies that n is even. Hence, the left hand side of 7 is equal to $\left(S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n)-1}^{(m)}\right) \vee S_{m-l_N(n)}^{(m)}$ and the right hand side of 7 is equal to $S_{m-l_N(n)-1}^{(m)} \vee S_{m-l_n(n)}^{(m)}$. If n is equal to 2 and m is equal to $N+1$, these two morphisms are equal to $S_0^{(N+1)} \vee S_1^{(N+1)}$. If the couple (n, m) is different from $(2, N+1)$, Relation 7 is a consequence of Lemma 4.3.1 point 2.

□

4.3.3. Lemma. *Let n and m be two integers such that $n \geq 2$ and $m \geq l_N(n+2)$. Letting*

$$T_{n,m} = S_{m-l_N(n+2)+1}^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)},$$

we have:

$$T_{n,m} \vee F_2^{n-1,m} = F_2^{n,m}.$$

Proof. From Lemma 4.1.3, we have

$$F_2^{n-1,m} = S_{m-l_N(n)}^{(m)} \vee \cdots \vee S_{m-N}^{(m)}, \ \text{and}$$
$$F_2^{n,m} = S_{m-l_N(n+1)}^{(m)} \vee \cdots \vee S_{m-N}^{(m)}.$$

The law $\vee$ being associative, it is sufficient to show:

$$T_{n,m} \vee S^{(m)}_{m-l_N(n)} = S^{(m)}_{m-l_N(n+1)} \vee \cdots \vee S^{(m)}_{m-l_N(n)}. \tag{8}$$

Assume that n is odd. We have $l_N(n+2) = l_N(n+1)+1$. Hence, the left hand side of 8 is equal to $S^{(m)}_{m-l_N(n+1)} \vee S^{(m)}_{m-l_N(n)}$. Moreover, $l_N(n+1)-l_N(n)$ is equal to $N-1$. Thus, Relation 8 is a consequence of Lemma 4.3.1 point 1.

Assume that n is even. We have $l_N(n+1) = l_N(n)+1$. Hence, the left hand side of 8 is equal to $\left(S^{(m)}_{m-l_N(n+2)+1} \wedge \cdots \wedge S^{(m)}_{m-l_N(n)-1)}\right) \vee S^{(m)}_{m-l_N(n)}$ and the right hand side of 8 is equal to $S^{(m)}_{m-l_n(n)-1} \vee S^{(m)}_{m-l_N(n)}$. Moreover, $l_N(n+2)-1-l_N(n)$ is equal to $N-1$. Thus, Relation 8 is a consequence of Lemma 4.3.1 point 2. $\square$

4.3.4. Proposition. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra. Assume that* $\mathbf{A}$ *admits a side-confluent presentation* $\langle X \mid R\rangle$*. Then, the presentation* $\langle X \mid R\rangle$ *satisfies the extra-condition if and only if for every integers* n *and* m *such that* $n \geq 1$ *and* $m \geq l_N(n+1)$*, we have:*

$$F_1^{n,m} \wedge \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right) = F_1^{n,m} \wedge F_2^{n,m}.$$

Proof. For every integers n and m such that $n \geq 1$ and $m \geq l_N(n+1)$, let

$$\begin{aligned} L_{n,m} &= F_1^{n,m} \wedge \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right), \\ R_{n,m} &= F_1^{n,m} \wedge F_2^{n,m}. \end{aligned}$$

Step 1. Assume that $n=1$. Fist, we show that:

$$L_{1,m} = F_1^{0,m}. \tag{9}$$

The kernel of $F_2^{0,m}$ is equal to $V^{\otimes m-1} \otimes J_1 = V^{\otimes m}$, that is, $F_2^{0,m}$ is equal to $0_{V^{\otimes m}}$. In particular, $F_1^{0,m} \vee F_2^{0,m}$ is equal to $F_1^{0,m}$. Moreover, the kernel of $F_1^{1,m}$ is equal to $I(R)_{m-1} \otimes V$ and the kernel of $F_1^{0,m}$ is equal to $I(R)_m$. The inclusion $I(R)_m \subset I(R)_{m-1} \otimes V$ implies that $F_1^{0,m}$ is smaller than $F_1^{1,m}$. Hence, Relation 9 holds.

Assume that $m = N$. The kernel of $F_1^{1,N}$ is equal to $I(R)_{N-1} \otimes V = \{0\}$, that is, $F_1^{1,N}$ is equal to $\mathrm{Id}_{V^{\otimes N}}$. In particular, $R_{1,N}$ is equal to $F_2^{1,N}$. Moreover, we have:

$$\begin{aligned} F_1^{0,N} &= {\theta_{X^{(N)}}}^{-1}\left(I(R)_N\right) \\ &= {\theta_{X^{(N)}}}^{-1}\left(R\right), \text{ and} \\ F_2^{1,N} &= {\theta_{X^{(N)}}}^{-1}\left(J_2\right) \\ &= {\theta_{X^{(N)}}}^{-1}\left(R\right). \end{aligned}$$

Thus $L_{1,N}$ and $R_{1,N}$ are equal.

Assume that $m \geq N+1$. From Lemma 4.1.3, we have:

$$\begin{aligned} F_1^{0,m} &= S_0^{(m)} \wedge \cdots \wedge S^{(m)}_{m-N}, \\ F_1^{1,m} &= S_0^{(m)} \wedge \cdots \wedge S^{(m)}_{m-N-1}, \\ F_2^{1,m} &= S^{(m)}_{m-N}. \end{aligned}$$

Thus, $R_{1,m}$ is equal to $F_1^{0,m}$. We conclude that Proposition 4.3.4 holds for $n=1$ and $m \geq N$.

Step 2. Assume that, $n \geq 2$ and $l_N(n+1) \leq m < l_N(n+2)$. From Lemma 4.1.2, $m - l_N(n)$ is smaller than $N-1$. Thus, the kernel of $F_1^{n,m}$ is equal to $\{0\}$, that is, $F_1^{n,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. In particular, $L_{n,m}$ is equal to $F_1^{n-1,m} \vee F_2^{n-1,m}$ and $R_{n,m}$ is equal to $F_2^{n,m}$.

From Lemma 4.1.3, we have:

$$\begin{aligned} F_1^{n-1,m} &= S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)}, \\ F_2^{n-1,m} &= S_{m-l_N(n)}^{(m)} \vee \cdots \vee S_{m-N}^{(m)}, \\ F_2^{n,m} &= S_{m-l_N(n+1)}^{(m)} \vee \cdots \vee S_{m-N}^{(m)}. \end{aligned}$$

Moreover, from Lemma 4.3.2, we have:

$$\left(S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)}\right) \vee S_{m-l_N(n)}^{(m)} = S_{m-l_N(n+1)}^{(m)} \vee \cdots \vee S_{m-l_N(n)}^{(m)}.$$

The law $\vee$ being associative, we deduce that Proposition 4.3.4 holds for every integers n and m such that $n \geq 2$ and $l_N(n+1) \leq m < l_N(n+2)$.

Step 3. Assume that $n \geq 2$ and $m \geq l_N(n+2)$. From Lemma 4.1.3, we have:

$$\begin{aligned} F_1^{n-1,m} &= S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)}, \text{ and} \\ F_1^{n,m} &= S_0^{(m)} \wedge \cdots \wedge S_{m-l_N(n+2)}^{(m)}. \end{aligned}$$

Thus, letting $T_{n,m} = S_{m-l_N(n+2)+1}^{(m)} \wedge \cdots \wedge S_{m-l_N(n+1)}^{(m)}$, we have:

$$F_1^{n-1,m} = F_1^{n,m} \wedge T_{n,m}.$$

The lattice generated by $S_0^{(m)}, \cdots, S_{m-N}^{(m)}$ being distributive, we have:

$$F_1^{n-1,m} \vee F_2^{n-1,m} = \left(F_1^{n,m} \vee F_2^{n-1,m}\right) \wedge \left(T_{n,m} \vee F_2^{n-1,m}\right).$$

Using the inequality $F_1^{n,m} \preceq \left(F_1^{n,m} \vee F_2^{n-1,m}\right)$, we deduce:

$$L_{n,m} = F_1^{n,m} \wedge \left(T_{n,m} \vee F_2^{n-1,m}\right).$$

From Lemma 4.3.3, $T_{n,m} \vee F_2^{n-1,m}$ is equal to $F_2^{n,m}$. Thus, Proposition 4.3.4 holds for every integers n and m such $n \geq 2$ and that $m \geq l_N(n+2)$. □

4.3.5. Theorem. *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra admitting an extra-confluent presentation* $\langle X \mid R \rangle$*. The left bound of* $\langle X \mid R \rangle$ *is a contracting homotopy for the Koszul complex of* $\mathbf{A}$.

Proof. Let ϕ be the endomorphism of $\mathrm{T}(V)$ which maps any element to its unique normal form for $\langle X \mid R \rangle$.

The presentation $\langle X \mid R \rangle$ is side-confluent. Thus, from Proposition 4.2.5, it is sufficient to show that for every integers n and m such that $n \geq 1$ and $m \geq l_N(n)$ we have:

$$(r_{n,m}) \quad F_1^{n,m} \wedge F_2^{n,m}{}_{|K_n^{(m)}} = F_1^{n-1,m} \vee F_2^{n-1,m}{}_{|K_n^{(m)}},$$

where $K_n^{(m)}$ is the vector space $\mathrm{im}\left(\phi_{|V^{\otimes m-l_N(n)}}\right) \otimes J_n$.

Assume that $l_N(n) \leq m < l_N(n+1)$. We show that $F_1^{n,m} \wedge F_2^{n,m}$ and $F_1^{n-1,m} \vee F_2^{n-1,m}$ are equal to $\mathrm{Id}_{V^{\otimes m}}$.

The hypothesis $l_N(n) \leq m < l_N(n+1)$ implies that $m - l_N(n)$ is smaller than $N-1$. In particular, the kernel of $F_1^{n,m}$ is equal to $\{0\}$, that is, $F_1^{n,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. Moreover, $F_2^{n,m}$ is also equal to $\mathrm{Id}_{V^{\otimes m}}$. Thus, the morphism $F_1^{n,m} \wedge F_2^{n,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. From Lemma 4.1.2, the morphism $F_1^{n-1,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$. Thus $F_1^{n-1,m} \vee F_2^{n-1,m}$ is equal to $\mathrm{Id}_{V^{\otimes m}}$ and Relation $(r_{n,m})$ holds.

Assume that $m \geq l_N(n+1)$. From Lemma 4.2.6 the operators $F_1^{n,m}$ and $F_1^{n-1,m} \vee F_2^{n-1,m}$ commute. We deduce from Relation 4a (see page 15):

$$F_1^{n,m} \wedge \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right) = \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right) F_1^{n,m}.$$

From Lemma 4.2.1, the image of $F_1^{n,m}$ is equal to $\mathrm{im}\left(\phi_{|V^{\otimes m-l_N(n)}}\right) \otimes V^{\otimes l_N(n)}$. Thus, $K_n^{(m)}$ is included in $\mathrm{im}\left(F_1^{n,m}\right)$. Hence, the restriction of $F_1^{n,m} \wedge \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right)$ to $K_n^{(m)}$ is equal to the restriction of $F_1^{n-1,m} \vee F_2^{n-1,m}$ to $K_n^{(m)}$. Moreover, the presentation $\langle X \mid R\rangle$ satisfies the extra-condition. Thus, from Proposition 4.3.4, $F_1^{n,m} \wedge \left(F_1^{n-1,m} \vee F_2^{n-1,m}\right)$ is equal to $F_1^{n,m} \wedge F_2^{n,m}$. Hence, Relation $(r_{n,m})$ holds. $\square$

5 Examples

In this section, we consider three examples of algebras which admit an extra-confluent presentation: the symmetric algebra, *monomial algebras satisfying the overlap property* and the enveloping algebra of the Heisenberg Lie algebra. For each of these examples we explicit the left bound constructed in Section 4.2.

5.1 The symmetric algebra

In this section we consider the symmetric algebra $\mathbf{A} = \mathbb{K}[x_1, \cdots, x_d]$ over d generators. This algebra admits the presentation $\langle X \mid R\rangle$, where the set X is equal to $\{x_1, \cdots, x_d\}$ and the set R is equal to $\{x_ix_j = x_jx_i,\ 1 \leq i \neq j \leq d\}$.

5.1.1. Extra-confluence. We consider the order $x_1 < \cdots < x_d$. The operator $S \in \mathrm{End}\left(V^{\otimes 2}\right)$ of the presentation $\langle X \mid R\rangle$ is defined on the basis $X^{(2)}$ by

$$S(x_ix_j) = \begin{cases} x_jx_i, & \text{if } i > j, \\ x_ix_j, & \text{otherwise.} \end{cases}$$

Let $w = x_ix_jx_k \in X^{(3)}$. If k is strictly smaller than j and i is strictly smaller than k, we have

$$\begin{aligned} \langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S\rangle^3 (w) &= \langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V\rangle^3 (w) \\ &= x_kx_jx_i. \end{aligned}$$

In the other cases the elements $\langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S\rangle^2 (w)$ and $\langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V\rangle^2 (w)$ are equal. In particular the two operators $\langle S \otimes \mathrm{Id}_V, \mathrm{Id}_V \otimes S\rangle^3$ and $\langle \mathrm{Id}_V \otimes S, S \otimes \mathrm{Id}_V\rangle^3$ are equal. Moreover, N is equal to 2. Thus, from Remark 2.3.3, the presentation $\langle X \mid R\rangle$ is extra-confluent. The normal form of a word $x_{i_1} \cdots x_{i_n}$ is equal to $x_{j_1} \cdots x_{j_n}$ where $\{j_1, \cdots, j_n\} = \{x_{i_1}, \cdots, x_{i_n}\}$ and $j_1 \leq \cdots \leq j_n$.

5.1.2. The Koszul complex of the symmetric algebra. The morphism $\partial_1 : \mathbf{A} \otimes V \longrightarrow \mathbf{A}$ is defined by $\partial_1(1_{\mathbf{A}} \otimes x_i) = \overline{x_i}$, for every $1 \leq i \leq d$. The morphism $\partial_2 : \mathbf{A} \otimes \overline{R} \longrightarrow \mathbf{A} \otimes V$ is defined by

$\partial_2(1_{\mathbf{A}} \otimes (x_j x_i - x_i x_j)) = \overline{x_j} \otimes x_i - \overline{x_i} \otimes x_j$, for every $1 \leq i < j \leq d$. If d is greater than 3, the vector space J_3 is spanned by the elements

$$\begin{aligned} r_{i_1<i_2<i_3} &:= x_{i_3}\left(x_{i_2}x_{i_1} - x_{i_1}x_{i_2}\right) - x_{i_2}\left(x_{i_3}x_{i_1} - x_{i_1}x_{i_3}\right) + x_{i_1}\left(x_{i_3}x_{i_2} - x_{i_2}x_{i_3}\right) \\ &= \left(x_{i_3}x_{i_2} - x_{i_2}x_{i_3}\right)x_{i_1} - \left(x_{i_3}x_{i_1} - x_{i_1}x_{i_3}\right)x_{i_2} + \left(x_{i_2}x_{i_1} - x_{i_1}x_{i_2}\right)x_{i_3}, \end{aligned}$$

where $1 \leq i_1 < i_2 < i_3 \leq d$. The morphism $\partial_3 : \mathbf{A} \otimes J_3 \longrightarrow \mathbf{A} \otimes \overline{R}$ maps the element $1_{\mathbf{A}} \otimes r_{i_1<i_2<i_3}$ to $\overline{x_{i_3}} \otimes \left(x_{i_2}x_{i_1} - x_{i_1}x_{i_2}\right) - \overline{x_{i_2}} \otimes \left(x_{i_3}x_{i_1} - x_{i_1}x_{i_3}\right) + \overline{x_{i_1}} \otimes \left(x_{i_3}x_{i_2} - x_{i_2}x_{i_3}\right)$.

Assume that d is greater than 4 and let n be an integer such that $3 \leq n \leq d-1$. We denote by I_n the set of sequences $i_1 < \cdots < i_n$ such that $1 \leq i_1$ and $i_n \leq d$. Assume that r_l is defined for every $l \in I_n$. For every $l = i_1 < \cdots < i_{n+1} \in I_{n+1}$ and every $1 \leq j \leq n+1$ we denote by l_j the element of I_n obtained from l removing i_j. Then, let

$$r_l = \sum_{j=0}^{n+1} (-1)^{-\eta(n+j)} x_{i_j} r_{l_j},$$

where $\eta : \mathbb{N} \longrightarrow \{-1, 1\}$ is defined by $\eta(k) = 1$ if k is even and $\eta(k) = -1$ if k is odd. For every $4 \leq n \leq d$, the vector space J_n is spanned by the elements r_l for $l \in I_n$. The map $\partial_n : \mathbf{A} \otimes J_n \longrightarrow \mathbf{A} \otimes J_{n-1}$ is defined by

$$\partial_n\left(1_{\mathbf{A}} \otimes r_l\right) = \sum_{j=1}^{n} (-1)^{-\eta(n-1+j)} \overline{x_{i_j}} \otimes r_{l_j}.$$

For every integer n such that $n \geq d+1$, J_n is equal to $\{0\}$.

5.1.3. The construction of h_1. Let m be an integer such that $m \geq 2$. Let $P_{1,m} = \left(F_1^{1,m}, F_2^{1,m}\right)$ be the reduction pair of bi-degree $(1, m)$ associated with $\langle X \mid R \rangle$. The morphisms $F_1^{1,m}$ and $F_2^{1,m}$ are defined by

$$F_1^{1,m}\left(x_{i_1} \cdots x_{i_m}\right) = \widehat{w} x_{i_m}, \text{ where } w = x_{i_1} \cdots x_{i_{m-1}}, \text{ and}$$

$$F_2^{1,m}\left(x_{i_1} \cdots x_{i_m}\right) = x_{i_1} \cdots x_{i_{m-2}} \widehat{w}, \text{ where } w = x_{i_{m-1}} x_{i_m}.$$

These morphisms satisfy the relation $\left\langle F_1^{1,m}, F_2^{1,m} \right\rangle^4 = \left\langle F_2^{1,m}, F_1^{1,m} \right\rangle^3$. Thus, we consider the $P_{1,m}$-representation of $\mathscr{A}_4$:

$$\begin{aligned} \varphi_{1,m} \colon \mathscr{A}_4 &\longrightarrow \operatorname{End}\left(V^{\otimes m}\right). \\ s_i &\longmapsto F_i^{1,m} \end{aligned}$$

The image of $\gamma_1 = (1 - s_2)(s_1 + s_1 s_2 s_1)$ through this morphism is equal to $F_1^{1,m} - F_2^{1,m} F_1^{1,m}$. Let $w x_{i_1} \in X^{(m)}$. Denoting by $\widehat{w} = w' x_{i_2}$, $\varphi_{1,m}\left(\gamma_1\right)\left(w x_{i_1}\right)$ is equal to $w'\left(x_{i_2} x_{i_1} - x_{i_1} x_{i_2}\right)$ if $i_2 < i_1$ and $\varphi_{1,m}\left(\gamma_1\right)\left(w x_{i_1}\right)$ is equal to 0, otherwise. Then, the map $h_1 : \mathbf{A} \otimes V \longrightarrow \mathbf{A} \otimes \overline{R}$ is defined by

$$h_1\left(\overline{w} \otimes x_{i_1}\right) = \begin{cases} \overline{w'} \otimes \left(x_{i_2} x_{i_1} - x_{i_1} x_{i_2}\right), \text{ if } i_2 < i_1, \\ 0, \text{ otherwise.} \end{cases}$$

5.1.4. The construction of h_2. Let m be an integer such that $m \geq 3$. Let $P_{2,m} = \left(F_1^{2,m}, F_2^{2,m}\right)$ be the reduction pair of bi-degree $(2, m)$ associated with $\langle X \mid R \rangle$. The morphisms $F_1^{2,m}$ and $F_2^{2,m}$ are defined by

$$F_1^{1,m}\left(x_{i_1} \cdots x_{i_m}\right) = \widehat{w} x_{i_{m-1}} x_{i_m}, \text{ where } w = x_{i_1} \cdots x_{i_{m-2}}, \text{ and}$$

$$F_2^{1,m}\left(x_{i_1}\cdots x_{i_m}\right)=\begin{cases} x_{i_1}\cdots x_{i_m-3}\left(r_{i_{m-2}<i_{m-1}<i_m}\right), \text{ if } i_{m-2}<i_{m-1}<i_m, \\ 0, \text{ otherwise.}\end{cases}$$

These morphisms satisfy the relation $\left\langle F_1^{2,m}, F_2^{2,m}\right\rangle^4 = \left\langle F_2^{2,m}, F_1^{2,m}\right\rangle^3$. Thus, we consider the $P_{2,m}$-representation of $\mathscr{A}_4$:

$$\begin{aligned}\varphi_{2,m}: \mathscr{A}_4 &\longrightarrow \operatorname{End}\left(V^{\otimes m}\right). \\ s_i &\longmapsto F_i^{2,m}\end{aligned}$$

The image of $\gamma_1 = (1-s_2)(s_1+s_1s_2s_1)$ through this morphism is equal to $F_1^{2,m} - F_2^{2,m}F_1^{2,m}$. Let $wx_{i_2}x_{i_1} \in X^{(m)}$. Denoting by $\widehat{w} = w'x_{i_3}$, $\varphi_{2,m}(\gamma_1)(wx_{i_2}x_{i_1})$ is equal to $w'r_{i_1<i_2<i_3}$ if $i_1<i_2<i_3$ and $\varphi_{2,m}(\gamma_1)(wx_{i_2}x_{i_1})$ is equal to 0 otherwise. Then, the map $h_2 : \mathbf{A}\otimes\overline{R} \longrightarrow \mathbf{A}\otimes J_3$ is defined by

$$h_2\left(\overline{w}\otimes\left(x_{i_2}x_{i_1}-x_{i_1}x_{i_2}\right)\right)=\begin{cases}\overline{w'}\otimes\left(r_{i_1<i_2<i_3}\right), \text{ if } i_1<i_2<i_3, \\ 0, \text{ otherwise.}\end{cases}$$

5.1.5. The construction of h_n. More generally, for every $\overline{w}\otimes r_{i_1<\cdots<i_n}$ we denote by $\widehat{w} = w'x_{i_{n+1}}$. The map $h_n : \mathbf{A}\otimes J_n \longrightarrow \mathbf{A}\otimes J_{n+1}$ is defined by

$$h_n\left(\overline{w}\otimes r_{i_1<\cdots<i_n}\right)=\begin{cases}\overline{w'}\otimes r_{i_1<\cdots<i_{n+1}}, \text{ if } i_1<\cdots<i_{n+1}, \\ 0, \text{ otherwise.}\end{cases}$$

5.1.6. Remark. The left bound family of $\langle X \mid R\rangle$ is the contracting homotopy constructed in the proof of [LV12, Proposition 3.4.8].

5.2 Monomial algebras satisfying the overlap property

In the section we consider the example from [Ber01, Proposition 3.8]. We consider a *monomial algebra* $\mathbf{A}$ over d generators: $X=\{x_1,\cdots,x_d\}$ and $R=\{w_1,\cdots,w_l\}$ is a set of words of length N. We assume that the presentation $\langle X\mid R\rangle$ satisfies the *overlap property*. This property is stated as follows:

5.2.1. The overlap property. For every integer n such that $N+2\leq n\leq 2N-1$ and for any word $w=x_{i_1}\cdots x_{i_n}$ such that $x_{i_1}\cdots x_{i_N}$ and $x_{i_{n-N+1}}\cdots x_{i_n}$ belong to R, all the sub-words of length N of w belong to R.

5.2.2. Extra-confluence. For any choice of order on X, the operator $S\in\operatorname{End}\left(V^{\otimes N}\right)$ of the presentation $\langle X\mid R\rangle$ is defined on the basis $X^{(N)}$ by

$$S(w)=\begin{cases}0, \text{ if } w\in R, \\ w, \text{ otherwise.}\end{cases}$$

As a consequence, for every integer m such that $1\leq m\leq N-1$, the operators $S\otimes\mathrm{Id}_{V^{\otimes m}}$ and $\mathrm{Id}_{V^{\otimes m}}\otimes S$ commute. Thus, the presentation $\langle X\mid R\rangle$ is side-confluent. Moreover, for monomial algebras, the extra-condition is equivalent to the overlap property. Thus, the presentation $\langle X\mid R\rangle$ is extra-confluent. The normal form of a word w is equal to 0 if w admits a sub-word which belongs to R, and w otherwise.

5.2.3. The Koszul complex of a monomial algebra. Let n be an integer such that $n\geq 2$. The vector space J_n is spanned by words w of length $l_N(n)$ such that every sub-word of length N of w belongs to R. The morphism $\partial_n : \mathbf{A}\otimes J_n \longrightarrow \mathbf{A}\otimes J_{n-1}$ maps $1_{\mathbf{A}}\otimes x_{i_1}\cdots x_{i_{l_N(n)}}$ to $\overline{w'}\otimes x_{i_{l_N(n)-l_N(n-1)+1}}\cdots x_{i_{l_N(n)}}$, where w' is equal to $x_{i_1}\cdots x_{i_{l_N(n)-l_N(n-1)}}$.

5.2.4. The contracting homotopy. Let n and m be two integers such that $m \geq l_n(n)$. Let $P_{n,m} = (F_1^{n,m}, F_2^{n,m})$ be the reduction pair of bi-degree (n, m) associated with $\langle X \mid R\rangle$. The operators $F_1^{n,m}$ and $F_2^{n,m}$ are defined by

$$F_1^{n,m}(x_{i_1}\cdots x_{i_m}) = \begin{cases} 0, \text{ if a sub-word of length } N \text{ of } x_{i_1}\cdots x_{i_{m-l_n(n)}} \text{ belongs to } R, \\ w, \text{ otherwise,} \end{cases}$$

and

$$F_2^{n,m}(x_{i_1}\cdots x_{i_m}) = \begin{cases} 0, \text{ if } x_{i_{m-l_N(n+1)+1}}\cdots x_{i_m} \in J_{n+1}, \\ w, \text{ otherwise.} \end{cases}$$

These operators commute. Thus, we consider the $P_{n,m}$-representation of $\mathscr{A}_2$:

$$\begin{aligned} \varphi_{n,m} \colon \mathscr{A}_2 &\longrightarrow \operatorname{End}\left(V^{\otimes m}\right). \\ s_i &\longmapsto F_i^{2,m} \end{aligned}$$

The image of $\gamma_1 = (1-s_2)s_1$ through this morphism is equal to $F_1^{n,m} - F_2^{n,m}F_1^{n,m}$. Let $w = x_{i_1}\cdots x_{i_m}$ be an element of $X^{(m)}$. If w is such that no sub-word of length N of $x_{i_1}\cdots x_{i_{m-l_n(n)}}$ belongs to R and if $x_{i_{m-l_N(n+1)+1}}\cdots x_{i_m}$ belongs to J_{n+1}, $\varphi_{n,m}(\gamma_1)(w)$ is equal to w. In the other cases $\varphi_{n,m}(\gamma_1)(w)$ is equal to 0. Then, the morphism $h_n : \mathbf{A}\otimes J_n \longrightarrow \mathbf{A}\otimes J_{n+1}$ is defined by

$$h_n\left(\overline{w}\otimes x_{i_{m-l_N(n)+1}}\cdots x_{i_m}\right) = \begin{cases} \overline{w'}\otimes x_{i_{m-l_N(n+1)+1}}\cdots x_{i_m}, \text{ if } x_{i_{m-l_N(n+1)+1}}\cdots x_{i_m} \in J_{n+1}, \\ 0, \text{ otherwise,} \end{cases}$$

where $w = x_{i_1}\cdots x_{i_{m-l_N(n)}}$ and $w' = x_{i_1}\cdots x_{i_{m-l_N(n+1)}}$.

5.3 The enveloping algebra of the Heisenberg Lie algebra

Let $\langle X \mid R\rangle$ be the presentation of Example 2.2.8. In this section we make explicit the left bound of $\langle X \mid R\rangle$. Recall that $X = \{x_1, x_2\}$ and $R = \{f_1, f_2\}$ where

$$\begin{aligned} f_1 &= x_2x_1x_1 - 2x_1x_2x_1 + x_1x_1x_2, \text{ and} \\ f_2 &= x_2x_2x_1 - 2x_2x_1x_2 + x_1x_2x_2. \end{aligned}$$

The acyclicty of the Koszul complex of any Yang-Mills algebra was proven in [CDV02, Theorem 1] and in [KVdB15, Section 6.3]. In this section, we propose an other proof for the enveloping algebra of the Heisenberg Lie algebra, based on the construction of an explicit contracting homotopy.

5.3.1. Extra-confluence. Recall that for the order $x_1 < x_2$, the operator $S \in \operatorname{End}\left(V^{\otimes 3}\right)$ of the presentation $\langle X \mid R\rangle$ is defined on the basis $X^{(3)}$ by

$$S(w) = \begin{cases} 2x_1x_2x_1 - x_1x_1x_2, \text{ if } w = x_2x_1x_1, \\ 2x_2x_1x_2 - x_1x_2x_2, \text{ if } w = x_2x_2x_1, \\ w, \text{ otherwise.} \end{cases}$$

Recall from Example 2.3.6 that this presentation is extra-confluent.

5.3.2. The Koszul complex of the enveloping algebra of the Heisenberg Lie algebra. The morphism $\partial_1 : \mathbf{A}\otimes V \longrightarrow \mathbf{A}$ is defined by $\partial_1(1_{\mathbf{A}}\otimes x_i) = \overline{x_i}$ for $i = 1$ or 2. The morphism $\partial_2 : \mathbf{A}\otimes\overline{R} \longrightarrow \mathbf{A}\otimes V$ is defined by

$$\begin{aligned} \partial_2(1_{\mathbf{A}}\otimes f_1) &= \overline{x_2x_1}\otimes x_1 - 2\overline{x_1x_2}\otimes x_1 + \overline{x_1x_1}\otimes x_2, \text{ and} \\ \partial_2(1_{\mathbf{A}}\otimes f_2) &= \overline{x_2x_2}\otimes x_1 - 2\overline{x_2x_1}\otimes x_2 + \overline{x_1x_2}\otimes x_2. \end{aligned}$$

The vector space $J_3 = \left(V \otimes \overline{R}\right) \cap \left(\overline{R} \otimes V\right)$ is the one-dimensional vector space spanned by

$$\begin{aligned} v &= x_2 f_1 + x_1 f_2 \\ &= f_2 x_1 + f_1 x_2. \end{aligned}$$

The morphism $\partial_3 : \mathbf{A} \otimes J_3 \longrightarrow \mathbf{A} \otimes \overline{R}$ is defined by

$$\partial_3 \left(1_A \otimes v\right) = \overline{x_2} \otimes f_1 + \overline{x_1} \otimes f_2.$$

For every integer n such that $n \geq 4$, the vector space J_n is equal to $\{0\}$.

5.3.3. The construction of h_1. Recall from Proposition 2.2.6 that the algebra $\mathbf{A}$ admits as a basis the set $\{\overline{w},\ w \in \langle X \rangle$ is a normal form$\}$. Thus, it is sufficient to define $h_1\left(\overline{w} \otimes x_i\right)$ for every normal form $w \in \langle X \rangle$ and $i = 1$ or 2.

Let m be an integer such that $m \geq 3$. Let $P_{1,m} = \left(F_1^{1,m}, F_2^{1,m}\right)$ be the reduction pair of bi-degree $(1, m)$ associated with $\langle X \mid R \rangle$. The morphisms $F_1^{1,m}$ and $F_2^{1,m}$ are defined by

$$F_1^{1,m}\left(x_{i_1} \cdots x_{i_m}\right) = \widehat{w} x_{i_m}, \text{ where } w = x_{i_1} \cdots x_{i_{m-1}}, \text{ and}$$

$$F_2^{1,m}\left(x_{i_1} \cdots x_{i_m}\right) = x_{i_1} \cdots x_{i_{m-3}} \widehat{w}, \text{ where } w = x_{i_{m-2}} x_{i_{m-1}} x_{i_m}.$$

These morphisms commute. Thus, we consider the $P_{1,m}$-representation of $\mathscr{A}_2$:

$$\begin{aligned} \varphi_{1,m} \colon \mathscr{A}_2 &\longrightarrow \operatorname{End}\left(V^{\otimes m}\right). \\ s_i &\longmapsto F_i^{1,m} \end{aligned}$$

The image of $\gamma_1 = (1 - s_2)s_1$ through this morphism is equal to $F_1^{1,m} - F_2^{1,m} F_1^{1,m}$.

Let w be a normal form such that the length of w is equal to $m-1$. The word wx_2 does not factorize on the right by $x_2x_1x_1$ or $x_2x_2x_1$. Thus, $\varphi_{1,m}\left(\gamma_1\right)\left(wx_2\right)$ is equal to 0. In particular, $h_1\left(\overline{w} \otimes x_2\right)$ is equal to 0 for every normal form $w \in \langle X \rangle$. If w does not factorize on the right by x_2x_1 or x_2x_2, $\varphi_{1,m}\left(\gamma_1\right)\left(wx_1\right)$ is equal to 0. Thus, $h_1\left(\overline{w} \otimes x_1\right)$ is equal to 0 for every normal form $w \in \langle X \rangle$ such that w does not factorize on the right by x_2x_1 or x_2x_2. If w can be written $w'x_2x_1$ (respectively $w'x_2x_2$), then $\varphi_{1,m}\left(\gamma_1\right)\left(wx_1\right)$ is equal to $w'\left(2x_1x_2x_1 - x_1x_1x_2\right)$ (respectively $w'\left(2x_2x_1x_2 - x_1x_2x_2\right)$). Thus, we have:

$$h_1\left(\overline{w} \otimes x_1\right) = \begin{cases} \overline{w'} \otimes \left(2x_1x_2x_1 - x_1x_1x_2\right), \text{ if } w = w'x_2x_1, \\ \overline{w'} \otimes \left(2x_2x_1x_2 - x_1x_2x_2\right), \text{ if } w = w'x_2x_2. \end{cases}$$

5.3.4. The construction of h_2. Recall from Proposition 2.2.6 that the algebra $\mathbf{A}$ admits as a basis the set $\{\overline{w},\ w \in \langle X \rangle$ is a normal form$\}$. Thus, it is sufficient to define $h_2\left(\overline{w} \otimes f_i\right)$ for every normal form $w \in \langle X \rangle$ and $i = 1$ or 2.

Let m be an integer such that $m \geq 4$. Let $P_{2,m} = \left(F_1^{2,m}, F_2^{2,m}\right)$ be the reduction pair of bi-degree $(2, m)$ associated with $\langle X \mid R \rangle$. The operator $F_1^{2,m}$ maps a word $w \in X^{(m)}$ to $\widehat{w_1} w_2$, where $w_1 \in \langle X \rangle$ and $w_2 \in X^{(4)}$ are such that $w = w_1 w_2$. The operator $F_2^{2,m}$ is equal to $\mathrm{Id}_{V^{\otimes m-4}} \otimes F$ where F is equal to $\theta_{X^{(4)}}^{-1}\left(J_3\right)$. The kernel of F is the one-dimensional vector space spanned by v. Thus, $F(\operatorname{lm}(v))$ is equal to $\operatorname{lm}(v) - v$, and for every $w \in X^{(4)} \setminus \{\operatorname{lm}(v)\}$, $F(w)$ is equal to w. Thus, F is defined on the basis $X^{(4)}$ by

$$F(w) = \begin{cases} 2x_2x_1x_2x_1 - x_2x_1x_1x_2 - x_1x_2x_2x_1 + 2x_1x_2x_1x_2 - x_1x_1x_2x_2, \text{ if } w = x_2x_2x_1x_1, \\ w, \text{ otherwise.} \end{cases}$$

The two operators $F_1^{2,m}$ and $F_2^{2,m}$ commute. Thus, we consider the $P_{2,m}$-representation of $\mathscr{A}_2$:

$$\begin{aligned} \varphi_{2,m} \colon \mathscr{A}_2 &\longrightarrow \operatorname{End}\left(V^{\otimes m}\right). \\ s_i &\longmapsto F_i^{2,m} \end{aligned}$$

The image of $\gamma_1 = (1 - s_2)s_1$ is equal to $F_1^{2,m} - F_2^{2,m}F_1^{2,m}$.

Let w be a normal form such that the length of w is equal to $m-1$. The word $x_2x_2x_1x_1$ does not occur in the decomposition of wf_2. Thus, $\varphi_{2,m}(wf_2)$ is equal to 0. In particular $h_2\left(\overline{w} \otimes f_2\right)$ is equal to 0 for every normal form $w \in \langle X \rangle$. If w does not factorize on the right by x_2, the word $x_2x_2x_1x_1$ does not occur in the decomposition of wf_1. Thus, $\varphi_{2,m}(wf_1)$ is equal to 0. In particular $h_2\left(\overline{w} \otimes f_1\right)$ is equal to 0 for every normal form $w \in \langle X \rangle$ such that w does not factorize on the right by x_2. Assume that w factorize on the right by x_2: $w = w'x_2$. Thus, $\varphi_{2,m}(wf_1)$ is equal to $w'\left(x_2x_2x_1x_1 - F(x_2x_2x_1x_1)\right)$. In this case we have

$$h_2\left(\overline{w} \otimes f_1\right) = \overline{w'} \otimes \left(x_2f_1 + x_1f_2\right).$$

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